SPECIAL KIND OF RELATION OF QUANTAL MODULES IN CONSTRUCTIONS WITH CATEGORIES AND FUNCTORS

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Abstract

Theory of categories is developed a lot using various mathematical objects. In too many cases objects within a category are interrelated structures of different kinds combined with different properties. It is seen that theory of categories has many applications within mathematics and many application outside mathematics. It is a powerful formal tool to formulate in an abstract way to many situations. As theory of lattices is combined with the theory of modules and yet more is combined with the theory of categories in applications in technology, it can be describe the way of a construction on them. We start from a model on the theory of categories. Modules over a ring are combined with the properties of homomorphism into the category theory where a new category can generate from a distinct object within it. It is an efficient way for objects to relate with important properties deriving by the same algebraic structure. On these categories one can operate with two special functors. Important properties related with a special quantale can be investigated into the theory of lattices. These are modules over special quantales. As these are special quantales we can continue with abstractions using categories and functors. Two kinds of categories can be constructed. The second one deriving from the first can have special construction of objects. In this category objects have another intern relation derive from the first category, yet we can operate through functors that deal with homomorphism. We tend to investigate other properties of quantal modules into the theory of categories and functors.

Keywords: *Category, homomorphism, module, quantale, sup-lattice.*

Introduction

Modules over a ring are combined with properties of homomorphism into the category theory where a new category can generate from a distinct object within it. On these categories one can operate with two special functors. All this can be imported into the theory of lattices as a way to connect important quantal modules.

1. Categories and functors

We see [4] that a *category* is a class of objects $\mathcal{C}$, for each pair A, B of which exists a set, whose elements are called morphisms from A to B, signed by $Hom_{\mathcal{C}}(A, B)$ for short $Hom(A, B)$; for each three objects (A, B, C) from $\mathcal{C}$ and for each pair of morphisms $f \epsilon Hom_{\mathcal{C}}(A, B)$, $g \epsilon Hom_{\mathcal{C}}(B, C)$ is defined their product (composition) $g \cdot f$ as a morphism in $Hom_{\mathcal{C}}(A, C)$, and

- $\forall (A,B), (A^{'}, B^{'}) \in \mathcal{C} \times \mathcal{C}$;
  $(A, B) \neq (A^{'}, B^{'}) \Rightarrow Hom_{\mathcal{C}}(A,B) \cap Hom_{\mathcal{C}}(A^{'}, B^{'}) = \emptyset$
  $\left(Hom_{\mathcal{C}}(A, B) \neq Hom_{\mathcal{C}}(A^{'}, B^{'}) \right)$;
- (*associative property*) $\forall f \epsilon Hom_{\mathcal{C}}(A, B), \forall g \epsilon Hom_{\mathcal{C}}(B, C), \forall h \epsilon Hom_{\mathcal{C}}(C, D)$,
  $(hg)f = h(gf)$;
- (*identity element*) $\forall A \in \mathcal{C}$, *there exists* $I_A \in Hom_{\mathcal{C}}(A, A)$ *such that*
  $\forall f \epsilon Hom_{\mathcal{C}}(A, B), \forall g \epsilon Hom_{\mathcal{C}}(B, A)$, $f \cdot I_A = f$ *and* $I_A \cdot g = g$.

So, a category $\mathcal{C}$ is caracterised by three esencial things:

1) it is a class of objects $Ob\mathcal{C}$;
2) for each pair (A, B) objects from the class there is a set of morphisms from A to B;
3) is defined the operation of composition $g \cdot f$ of two morphisms f, g such that $codom\, f = dom\, g$.

Given a category as above, by every object of it, we can construct a new one.
Let $\mathcal{C}$ be a category and let A be an object of it. We can construct a new category $\mathcal{C}_A$ in this way: the objects of $\mathcal{C}_A$ are $Hom_{\mathcal{C}}(A, X)$ and the morphisms from $Hom_{\mathcal{C}}(A, X)$ in $Hom_{\mathcal{C}}(A, Y)$ in $\mathcal{C}_A$ are the morphism $h: X \to Y$ of $\mathcal{C}$ such as, for all $f: A \to X$ and for all $g: A \to Y$, the diagram in figure 1 is commutative.

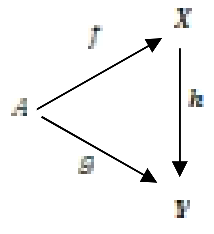

Fig. 1. A morphism h in $\mathcal{C}_A$

In a similar way, from $\mathcal{C}$ can be constructed another category $\mathcal{C}^A$ defined this way: the objects of $\mathcal{C}^A$ are the set $Hom_{\mathcal{C}}(X,A)$, morphisms of $\mathcal{C}^A$ from $Hom_{\mathcal{C}}(X,A)$ to $Hom_{\mathcal{C}}(Y,A)$ are the morphisms $h: X \to Y$ of $\mathcal{C}$ such as, for all $f: X \to A$ and for all $g: Y \to A$, the diagram in figure 2

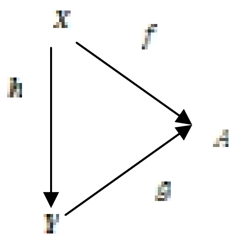

Fig. 2. A morphism h in $\mathcal{C}^A$

is commutative.
Let now bring another important component.
Also in [4], we find the concept of functor.

Functors associate to every object from a category an object of the same category o from another one, and not only. There are two general type of functors:

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be categories.

A *covariant functor* from category $\mathcal{C}_1$ to category $\mathcal{C}_2$, is a rule $\mathcal{F}$, that associates to each object $A \in \mathcal{C}_1$ an object $\mathcal{F}(A) \in \mathcal{C}_2$ and to each morphism $f \in Hom_{\mathcal{C}_1}(A,B)$ of $\mathcal{C}_1$ a morphism $\mathcal{F}(f)$: $\mathcal{F}(A) \to \mathcal{F}(B)$ from $Hom_{\mathcal{C}_2}(\mathcal{F}(A),\mathcal{F}(B))$ such that*:*

1. $\forall A \in \mathcal{C}_1, \mathcal{F}(I_A) = I_{\mathcal{F}(A)}$;
2. $\forall f \in Hom_{\mathcal{C}_1}(A,B), \forall g \in Hom_{\mathcal{C}_1}(B,C)$,is defined $\mathcal{F}(gf)$ such that $\mathcal{F}(gf) = \mathcal{F}(g)\mathcal{F}(f)$.

A contravariant functor from category $\mathcal{C}_1$ to category $\mathcal{C}_2$, is a rule $\mathcal{F}$ that associates to each object $A \in \mathcal{C}_1$ an object $\mathcal{F}(A) \in \mathcal{C}_2$ and to each morphism $f \in Hom_{\mathcal{C}_1}(A,B)$ of $\mathcal{C}_1$ a morphism $\mathcal{F}(f)$: $\mathcal{F}(B) \to \mathcal{F}(A)$ from $Hom_{\mathcal{C}_2}(\mathcal{F}(B),\mathcal{F}(A))$ such that:

1. $\forall A \in \mathcal{C}_1, \mathcal{F}(I_A) = I_{\mathcal{F}(A)}$;
2. $\forall f \in Hom_{\mathcal{C}_1}(A,B), \forall g \in Hom_{\mathcal{C}_1}(B,C)$, is defined $\mathcal{F}(gf)$ such that $\mathcal{F}(gf) = \mathcal{F}(f)\mathcal{F}(g)$.

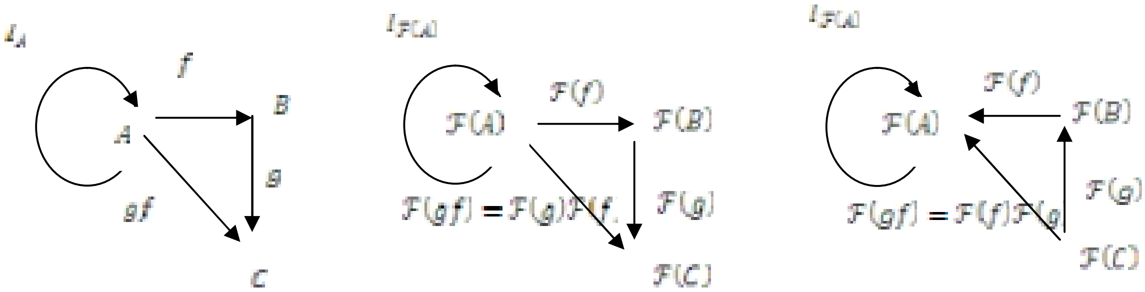

Fig. 3. Objects and morphisms in a category, a covariant, a contravariant functor

2. Sup-lattice, quantales, quantal-modules and their homomorphisms

A sup-lattice ([1],[3]) is a poset $\langle L, \leq \rangle$ which admits arbitrary joins.
Let Q be a non empty set in which is defined a partial order $\leq$ and a multiplication $\cdot : Q \times Q \to Q$ with a unit element e.

Definition 1. A *quantal* is an algebraic structure $Q = (Q, \vee, \cdot, \perp)$ such that:

1. $\langle Q, \vee, \perp \rangle$ is a sup-lattice;
2. $(Q, \cdot)$ is a monoid;
3. $\forall x \in Q, \{y_i\}_{i\in I} \subseteq Q, x \cdot \vee_{i\in I} y_i = \vee_{i\in I}(x \cdot y_i), \vee_{i\in I} y_i \cdot x = \vee_{i\in I}(y_i \cdot x)$.

Q is said to be commutative if so is the multiplication.

Let M be a non empty set in which is defined a partial order $\leq$ and Q is a quantale. It is also defined in M the algebraic operation denoted by $\cdot$ that maps $Q \times M$ to M. This operation is called the left multiplication in M with elements from Q.

Definition 2. *A left Q-module* is the algebraic structure $(M; \cdot)$ such that:

1. $\langle M, \vee, \perp \rangle$ is a sup-lattice;
2. $\forall (q_1, q_2, m) \in Q^2 \times M, (q_1 \cdot q_2) \cdot m = q_1 \cdot (q_2 \cdot m)$;
3. $\forall q \in Q, \{m_i\}_{i\in I} \subseteq M, q \cdot {}^M\vee_{i\in I} m_i = {}^M\vee_{i\in I} q \cdot m_i$;
4. $\forall \{q_i\}_{i\in I} \subseteq Q, m \in M, ({}^Q\vee_{i\in I} q_i) \cdot m = {}^M\vee_{i\in I} q_i \cdot m$;
5. $\forall m \in M, e \cdot m = m$.

In a similar way can be defined the right Q-module. Shortly, we can denote the left Q-module (the right Q-module) with ${}_QM, (M_Q)$.
If a Q-module is a left and right Q-module it is simply called a Q-module.
Let M and N be Q-modules.

Definition 3. *A homomorphism* (*Q-homomophism*) of Q-module M to Q-module N is a map $f: M \to N$ such that:

1. $f(\vee_{i\in I} m_i) = \vee_{i\in I} f(m_i), \forall \{m_i\}_{i\in I} \subseteq M$;
2. $f(q \cdot m) = q \cdot f(m), \forall q \in Q, m \in M_1$, ($f(q \cdot m) = q \cdot f(m), \forall q \in Q, m \in M_1$).

The set of all Q-homomorphism from M to N is denoted by $Hom_Q(M,N)$.
Let h_i, $i \in I$ be maps from Q-module M to Q-module N.

Definition 4. *Union of* h_i, $i \in I$, denoted by $\sqcup_{i\in I} h_i$, is called a map $\sqcup_{i\in I} h_i : M \to N$, defined by
$(\sqcup_{i\in I} h_i)(x) = \vee_{i\in I} h_i(x)$, for all $x \in M$.

LEMMA 1. *If $h_i \in Hom_Q(M,N), i \in I$, than $\sqcup_{i\in I} h_i$ is a Q-homomorphism of M in N.* [6]
In a similar way can be defined the right multiplication $f \cdot q$.

LEMMA 2. *If the map $f \in Hom_Q(M,N)$, and Q is a commutative quantale, than $q \cdot f$ $(f \cdot q) \in Hom_Q(M,N)$, for all $q \in Q$.* [6]

Definition 5. Associating to all the tupples $(q,f) \in Q \times Hom_Q(M,N)$, when Q is a commutative quantale, the homomorphism $q \cdot f \in Hom_Q(M,N)$, we have

$$\cdot : Q \times Hom_Q(M,N) \to Hom_Q(M,N),$$

that is called the *left multiplication* in $Hom_Q(M,N)$ with elements from commutative quantale Q.

Proposition. *If quantal is commutative and , are -modules, than the algebra is a left -module (right -module).* [6]

3. Category of quantale-modules and category from a quantale-module

Let Q be a quantale and let we considerate the set of all modules over the quantale Q.
By the properties of quantale-modules we see that, for the Q-module M_1, M_2
1) exist $Hom_Q(M_1,M_2)$ of all Q-homomorphism from M_1 to M_2;
2) for all tupples (M_1,M_2,M_3) of Q-modules and for all pairs of Q-homomorphisms $f \epsilon Hom_Q(M_1,M_2)$, $g \epsilon Hom_Q(M_2,M_3)$ is defined their multiplication (the composition) $g \cdot f$ in $Hom_Q(M_1,M_3)$, as a composition of Q-homomorphisms (a usual map composition).
In these conditions, we have:

- $\forall (M_1,M_2),(M_1^{'},M_2^{'}) \in C \times C$;
  $(M_1,M_2) \neq (M_1^{'},M_2^{'}) \Rightarrow Hom_C(M_1,M_2) \cap Hom_C(M_1^{'},M_2^{'}) = \emptyset$
  $(Hom_C(M_1,M_2) \neq Hom_C(M_1^{'},M_2^{'}))$
- (*associative property*)
  $\forall f \epsilon Hom_C(M_1,M_2), \forall g \epsilon Hom_C(M_2,M_3), \forall h \epsilon Hom_C(M_3,M_4)$, $(hg)f = h(gf)$;
- (*identity element*) $\forall M_1 \in C$ *there exists* $I_{M_1} \in Hom_C(M_1,M_1)$ *such that*
  $\forall f \epsilon Hom_C(M_1,M_2), \forall g \epsilon Hom_C(M_2,M_1)$, $f \cdot I_{M_1} = f$ *and* $I_{M_1} \cdot g = g$.

As a result of this the set of all Q-homomorphisms is a category.
Given a quantale Q, we denote $\mathcal{M}_Q^l$ and $\mathcal{M}_Q^r$ the categories whose objects are, respectively, the left Q-modules and the right Q-modules, and whose morphisms are the Q-module homomorphisms. If Q is a commutative quantale, than $\mathcal{M}_Q^l$ and $\mathcal{M}_Q^r$ are the same, so we denote this category by $\mathcal{M}_Q$.
With an object M of the category $\mathcal{M}_Q$ we can construct a new category $\mathcal{M}_{Q_M}$, as above. So, the objects are the sets of morphisms $Hom_{\mathcal{M}_Q}(M,X)$ between M and X in $\mathcal{M}_Q$, and the morphisms from $Hom_{\mathcal{M}_Q}(M,X)$ in $Hom_{\mathcal{M}_Q}(M,Y)$ are the morphism $h: X \to Y$, from $Hom_{\mathcal{M}_Q}(X,Y)$, in $\mathcal{M}_Q$ such that, for all $f: M \to X$ and for all $g: M \to Y$, the diagram in figure 4 is commutative

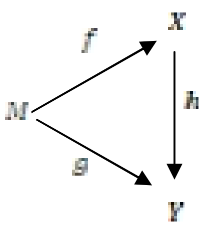

Fig. 4. A morphism h in $\mathcal{M}_{Q_M}$

We see that, the set of morphisms $Hom_{\mathcal{M}_Q}(M,X)$, in $\mathcal{M}_Q$ is the set $Hom_Q(M,X)$ of Q-homomorphisms from M to X. So, the object $Hom_{\mathcal{M}_Q}(M,X)$ of the category $\mathcal{M}_{Q_M}$ is the set $Hom_Q(M,X)$. The set of morphisms from the object $Hom_Q(M,X)$ to the object $Hom_Q(M,Y)$ in $\mathcal{M}_{Q_M}$ is $Hom_{\mathcal{M}_{Q_M}}(Hom_Q(M,X), Hom_Q(M,Y))$.

5. A connection between distinct quantal modules

Let Q be a commutative quantale and $\mathcal{M}_Q$ the category of Q-modules, where M is a fixed Q-module in $Ob\mathcal{M}_Q$, and let Ag be the category ob abelian groups. The set $Hom(M,N)$ is a Q-modul for every Q-module $N \in Ob\mathcal{M}_Q$, so it is an abelian group. Let $\mathcal{F}_M$ be a rule that asociate 1) to every Q-module $N \in Ob\mathcal{M}_Q$ the abelian group $Hom_R(M,N)$,

$$\mathcal{F}_M(N) = Hom_R(M,N), \forall N \in Ob\mathcal{M}_Q$$

2) to every Q-homomorphism $f: N \to N'$ of Q-modules $N, N' \in Ob\mathcal{M}_Q$ the morphism $\mathcal{F}_M(f): \mathcal{F}_M(N) \to \mathcal{F}_M(N')$, by $\mathcal{F}_M(f): h \mapsto f \cdot h$, $\forall h \in Hom_R(M,N)$. $\mathcal{F}_M$ has the properties

1. $\mathcal{F}_M(I_N): h \mapsto I_N \cdot h = h \Rightarrow \mathcal{F}_M(I_N) = 1_{\mathcal{F}_M(N)}$
2. $\mathcal{F}_M(g \cdot f)(h) = (g \cdot f) \cdot h = g \cdot (f \cdot h)$

$$= (\mathcal{F}_M(g))(f \cdot h) = (\mathcal{F}_M(g))\big((\mathcal{F}_M(f))(h)\big)$$

$$= \big((\mathcal{F}_M(g)) \cdot (\mathcal{F}_M(f))\big)(h) \Rightarrow$$

$$\mathcal{F}_M(g \cdot f) = \mathcal{F}_M(g) \cdot \mathcal{F}_M(f)$$

By the definition above $\mathcal{F}_M$ is a covariant functor from the category $\mathcal{M}_Q$ in Ag. The functor $\mathcal{F}_M$ is signed by $Hom_{\mathcal{M}_Q}(M; \ )$.

In a similar way can be constructed the contravariant functor $Hom_{\mathcal{M}_Q}(\ ; M)$.

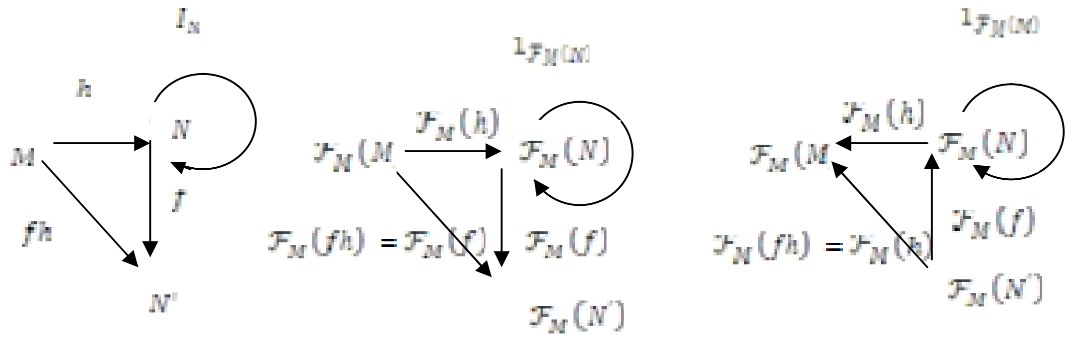

Fig. 5. Objects and morphisms in a category, the covariant, and contravariant Hom-functor $\mathcal{F}_M$

Conclusion

The relation of a distinct quantal module with other quantal modules can be investigated into another category still preserving the particular role of homomorphisms of quantal modules.

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