THE FRACTAL CONSTRUCTION BY ITERATED FUNCTION SYSTEMS

Elvira Goga

Polytechnic University of Tirana, Sheshi Nënë Tereza, Tirana, elvira_goga@yahoo.com

Abstract

This paper presents the theory and algorithms for computing fractals from Iterated Function Systems (IFS). The algorithms presented are the Deterministic Algorithm and the Random Iteration Algorithm in the space $\mathbb{R}^n$ ($n = 2$) with the Hausdorff metric. The Hausdorff metric is the most natural metric in comparing objects in an ideal geometric space, like the fractal objects illustrated in the paper. These algorithms will illustrate through visual means. The fundamental property of an Iterated Function System is that each IFS determines a unique attractor, which is typically a fractal. Any set of affine transformations and an associated set of probabilities determines an Iterated Function System. These probabilities play an important role in the computation of images of the attractor of an Iterated Function System using the Random Iteration Algorithm. They play no role in the Deterministic Algorithm. One of the advantages of using an Iterated Function System is that the dimension of the attractor is often relatively easy to estimate in terms of the defining contractions. A dimension contains much informacion about the geometrical properties of a set. This paper discusses about the Hausdorff dimension and box dimension of the attractor of an Iterated Function System consisting of contractions that are similarities. Finally, this paper gives construction of some fractals by Iterated Function Systems and finds the Hausdorff and box dimensions of some self-similar fractals.

***Keywords*:** fractal, iteration, deterministic algorithm, random algorithm, dimension

1. Introduction

The fractal, as a mathematical phenomenon, dates back to the Weierstrass nowhere differentiable continuous function (Figure 7), to the classic Cantor set (Figure 8), to the Hilbert space filling curve (Figure 9), and even beyond. The term fractal, which means broken or irregular fragments, was originally coined by Mandelbrot[1] to describe a family of complex shapes that possess an inherent self-similarity or self-affinity in their geometrical stucture. Fractal geometry provides a general framework for the study of such irregular sets. Fractal geometry can be used to make precise models of physical structures from ferns to galaxies[2]. A fractal is a geometrical object characterized by two fundamental properties: self-similarity and Hausdorff-Besicovitch dimension. A self-similar object is exactly or approximately similar to a part of itself and that can be continuously subdivided in parts each of which is a reduced-scale copy of the whole. The fractal dimension of a set is a number that tells how densely the set occupies the metric space which it lies. The Hausdorff-Besicovitch dimension of bounded subset of $\mathbb{R}^n$ is another real number that can be used to characterize the geometrical complexity of bounded subsets of $\mathbb{R}^n$.

Today's computers not only enable us to view fractals as amazing mathematical objects but also furnish new ways to model the real world[3]. A fractal can be generated by an algorithm because of self-similarity on every scale. Iterated function systems (IFS) represent an extremely versatile method for conveniently generating a wide variety of useful fractal structures [4][2].

2. The fractal theory and applications

Fractal geometry offers very important tools for describing and analyzing irregularity which can be classified as new regularity, seemingly random but with precise internal organization[5]. Fractal geometry explicitly uses the concept of observation scale in building of an object. An object is generated by repeating a process using a sequence of scales. However, all scales in a fractal are considered equivalent and objects at different scales are indistinguishable. This feature of fractals, called 'self-similarity', is perhaps their most important property. Many sets may be constructed using recursive procedures. For example the Sierpinski triangle is obtained by repeatedly removing equilateral triangles from an initial equilateral triangle of unit side-length (Figure 1)

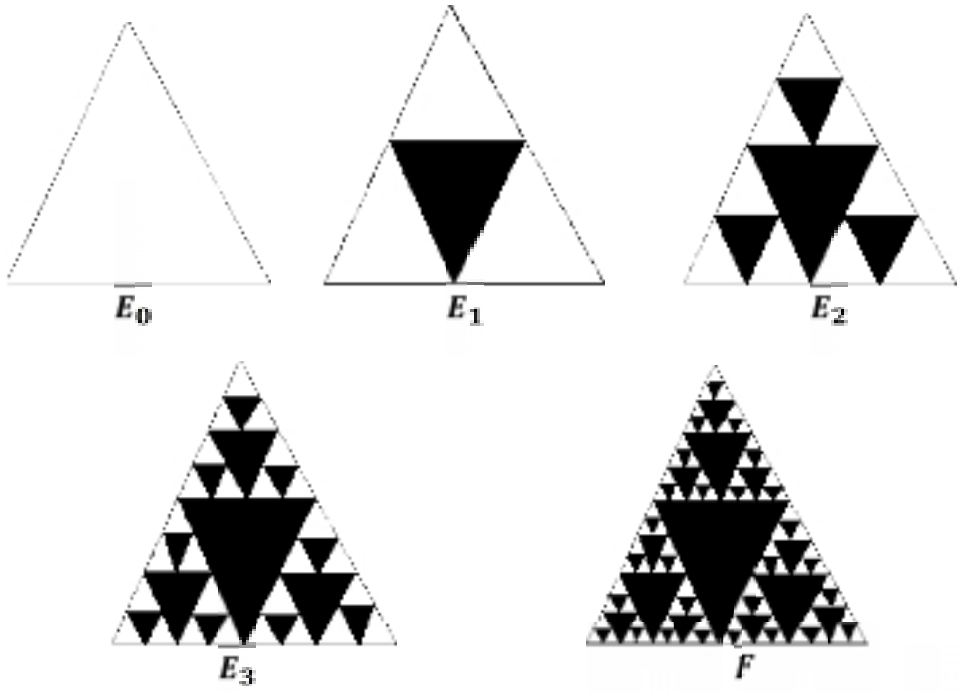

Figure 1. Construction of the Sierpinski triangle ($dim_H(F) = dim_B(F) = \frac{\log 3}{\log 2}$)

The notion of dimension is central to fractal geometry. The definition of Hausdorff dimension is the oldest and probably the most important. The advantage of Hausdorff dimension can be defined for any set. The disadvantage of Hausdorff dimension is that in many cases it is hard to calculate by computational methods. In mathematics, no universal definition of fractal dimension exists and several definitions of fractal dimension may lead to different results for the same object.

Figure 2 shows that the shape can be decomposed into $N = 3$ pieces, each scaled by a factor of $r = \frac{1}{2}$, so similarity dimension (the fractal dimension) can be calculated by the equation:

$$D = \frac{\log(N)}{\log(\frac{1}{r})} = \frac{\log(3)}{\log(\frac{1}{\frac{1}{2}})} = \frac{log3}{log2} \approx 1.589$$

Figure 2. Sierpinski triangle

Box-counting dimension is one of the most widely used dimensions. Its popularity is largely due to its relative case of mathematical calculation and empirical estimation[6].

Let F be any non-empty bounded subset of $\mathbb{R}^n$ and let $N_\delta(F)$ be the smallest number of sets of diameter at most δ which can cover F. The box-counting dimension of F is given by the formula:

$$dim_B(F) = \lim_{\delta \to 0} \frac{logN_\delta(F)}{\log(\frac{1}{\delta})} = \lim_{\delta \to 0} \frac{logN_\delta(F)}{-\log(\delta)}.$$

To find the box dimension of a plane set F we draw a mesh of squares or boxes of side δ and count the number $N_\delta(F)$ that overlap the set for various small δ. The dimension is logarithmic rate at which $N_\delta(F)$ increases as $\delta \to 0$, and may be estimated by the gradient of the graph of $logN_\delta(F)$ against $-log\delta$.

Figure 3 shows the box-counting method for the Koch curve. Covering the Koch curve with smaller and smaller boxes, from the relation $N(r_n) = 3 \cdot 4^{n-1}$, we can compute the exact value of the Koch curve:

$$D_B = \lim_{n \to \infty} \frac{\log N(r_n)}{\log(\frac{1}{r_n})}$$

$$= \lim_{n \to \infty} \frac{\log N((\frac{1}{3})^n)}{\log(\frac{1}{(\frac{1}{3})^n})}$$

$$= \lim_{n \to \infty} \frac{\log(3 \cdot 4^{n-1})}{\log(3^n)}$$

$$= \frac{Log(4)}{\log(3)} \approx 1.261$$

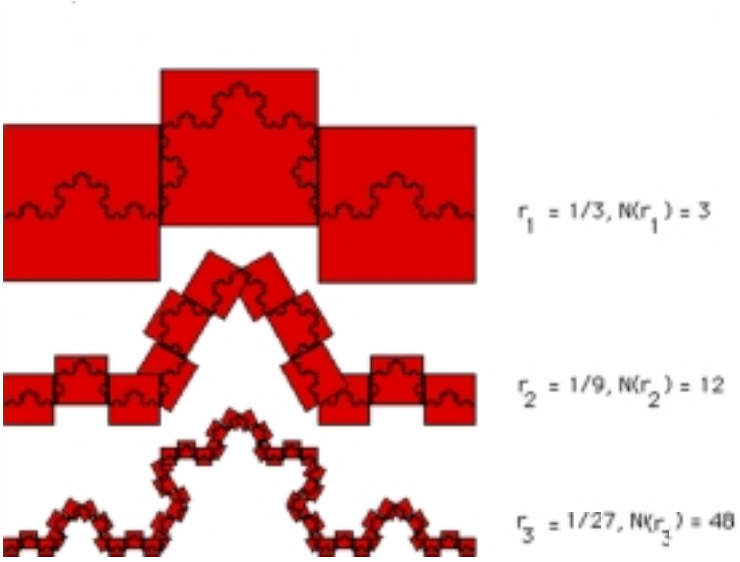

Figure 3. Covering the Koch curve with box size $r_1 = \frac{1}{3}; r_2 = \frac{1}{9}; r_3 = \frac{1}{27}$

One of the advantages of using an iterated function system (IFS) is that the dimension of the attractor is often relatively easy to calculate in terms of the defining contractions.
Let X be a closed subset of $\mathbb{R}^n$, often $X = \mathbb{R}^n$. A mapping $S: X \to X$ is called a contraction on X if there is a number c with $0 < c < 1$ such that $|S(x) - S(y)| \leq c|x - y|$ for all $x, y \in X$. If equality holds, $|S(x) - S(y)| = c|x - y|$, then S transforms sets into geometrically similar sets, and S is called contracting similarity.
A finite family of contractions $S = \{S_1, S_2, .., S_m\}$, with $m \geq 2$, is called an iterated function system or IFS. A non-empty compact subset F of X is called an attractor (or deterministic fractal) for the IFS if: $F = \bigcup_{i=1}^{m} S_i(F)$.
The fundamental property of an iterated function system is that it determines a unique attractor, which usually is a fractal.
If $S_1, S_2, .., S_m: \mathbb{R}^n \to \mathbb{R}^n$ are similarities ($|S_i(x) - S_i(y)| \leq c_i|x - y|$ where $0 < c_i < 1$, c_i is called ratio of S_i), each S_i transforms subsets of $\mathbb{R}^n$ into geometrically similar sets. The attractor of such a collection of similarities is called a self-similar set, being a union of a number of smaller similar copies of itself. See figures 1 and 4.

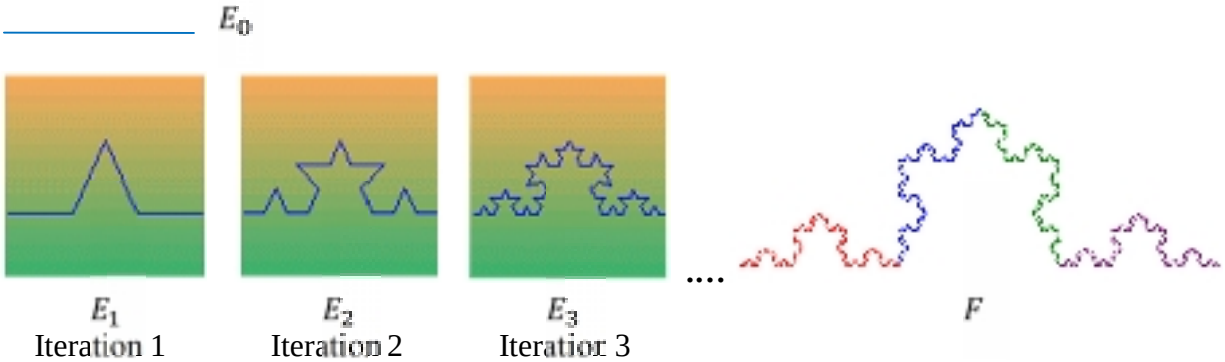

Figure 4. Construction of the von Koch curve F. At each stage, the middle third of each interval is replaced by the other two sides of an equilateral triangle.

A self-similar set F has Hausdorff and box dimensions equal to the value of d satisfying

$$\sum_{i=1}^{m} c_i{}^{d} = 1.$$

A transformation $S: \mathbb{R}^n \to \mathbb{R}^n$ is called an affine transformation if is of form $S(x) = T(x) + b$ where T is a linear transformation on $\mathbb{R}^n$ (represented by an $n \times n$ matrix) and b is a vector in $\mathbb{R}^n$. If an IFS consists of affine contractions $\{S_1, S_2, \ldots, S_m\}$ on $\mathbb{R}^n$, the attractor F is called a self-affine set. Two algorithms for computing fractals from Iterated Function Systems (IFS): the deterministic algorithm and the random iteration algorithm , will illustrate through visual means. We restrict attention to IFS of form $\{\mathbb{R}^2; S_i: i = i, 2, \ldots, m\}$,where each mapping is an affine transformation. An affine transformation, S, defined by $S\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$, where $a, b, c, d, e, and\ f$ are real number, such that $a, b, c\ and\ d$ control rotation and scaling, while $e\ and\ f$ control linear translation.

The deterministic algorithm is based on the idea of directly computing a sequence of sets $\{E_n = S^{\circ n}(F)\}$ starting from an initial set E_0. An IFS $S = \{S_1, S_2, \ldots, S_m\}$with probabilities $P = \{p_1, p_2, \ldots, p_m\}$ is an IFS with a positive number associated to each transformation, the total sum of the probabilities is 1 ($p_i > 0$ for all $i = 1, 2, \ldots, m$ and $\sum_{i=1}^{m} p_i = 1$). These probabilities play an important role in the computation of images of the attractor of an IFS using the random iteration algorithm. They play no role in the deterministic algorithm.

We illustrate the deterministic algorithm for an IFS whose attractor is a Koch curve, through Figures 5 and 4.

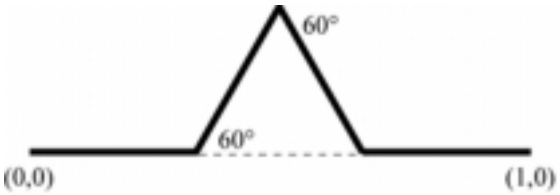

Figure 5. The first iteration of the Koch curve

The first iteration for the Koch curve consists of taking four copies of the horizontal line segment, each scaled by $r = \frac{1}{3}$. Two segments must be rotated by 60°, one counterclockwise and one clockwise. Along with the required translations, this yields the following IFS:

$$S_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \text{ scale by } r$$

$$S_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{6} & \frac{-\sqrt{3}}{6} \\ \frac{\sqrt{3}}{6} & \frac{1}{6} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix}, \text{ scale by } r, \text{ rotate by } 60^\circ$$

$$S_3\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{6} & \frac{\sqrt{3}}{6} \\ \frac{-\sqrt{3}}{6} & \frac{1}{6} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{6} \end{pmatrix}, \text{ scale by } r, \text{ rotate by } -60^\circ$$

$S_4\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\frac{1}{3} & 0\\ 0 & \frac{1}{3}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}\frac{2}{3}\\ 0\end{pmatrix}$, scale by r. The fixed attractor of this IFS is the Koch curve (F in Figure 4).

Figure 6 shows the result of running the random iteration algorithm for increasing numbers of iterations whose attractor is a fern.

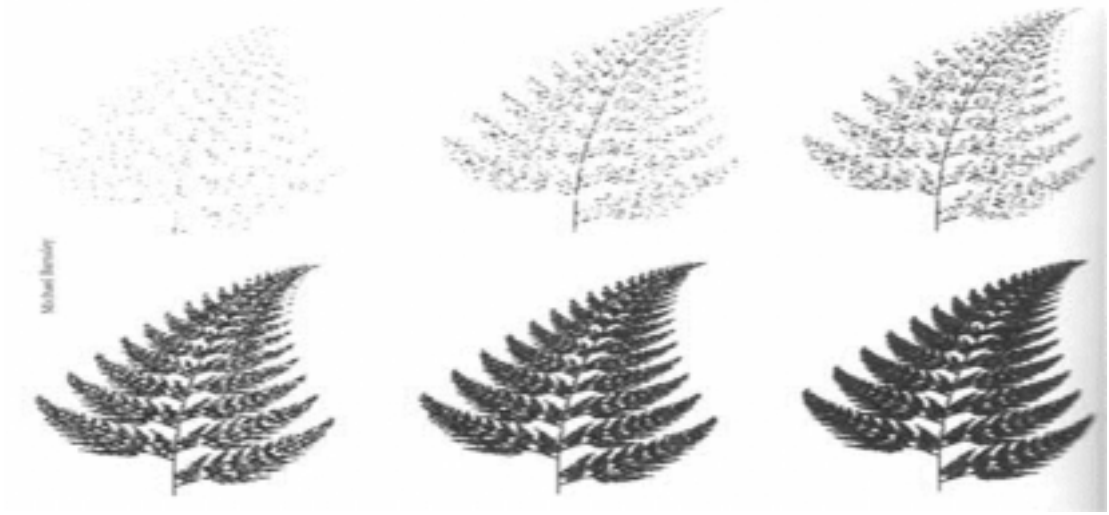

Figure 6. The result of running the random iteration algorithm for increasing numbers of iterations. The randomly dancing point starts to suggest the structure of the attractor of the IFS given in Table 1.

Table 1. IFS code for fern

S	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	p
1	0	0	0	0.16	0	0	0.01
2	0.85	0.04	-0.04	0.85	0	1.6	0.85
3	0.2	-0.26	0.23	0.22	0	1.6	0.07
4	-0.15	0.28	0.26	0.24	0	0.44	0.07

3. Conclusion

This paper presented a short overview of the Iterated Function Systems (IFS), Two algorithms for computing fractals from Iterated Function Systems have been illustrated through visual means. The self-similar nature of many fractals comes from the prescribed algorithm for the construction of the fractal and can be viewed as a transformation of some metric space. For example, the Cantor set can be viewed as the contraction and translation of an interval on the real line. Finally, this paper gives construction of some fractals by Iterated Function Systems and finds the Hausdorff and box dimensions of some self-similar fractals.

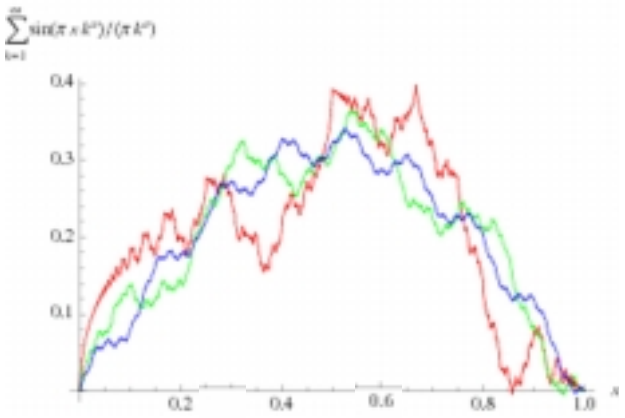

Figure 7. Weierstrass function. The plots show $f_a(x)$ for $a = 2$(red), $a = 3$(green) and 4(blue),

$$f_a(x) = \sum_{k=1}^{\infty} \frac{\sin(\pi k^a x)}{\pi k^a}$$

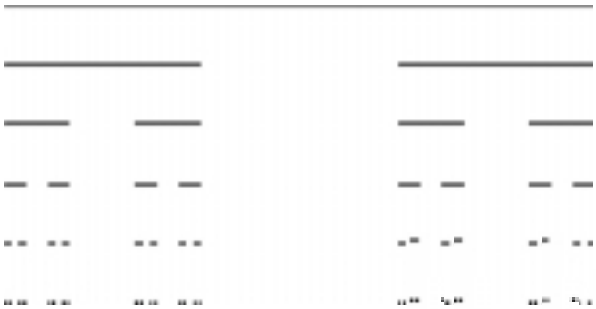

Figure 8. The first few stages $C_0, C_1, \ldots, C_5$ in the construction of middle-thirds Cantor set C

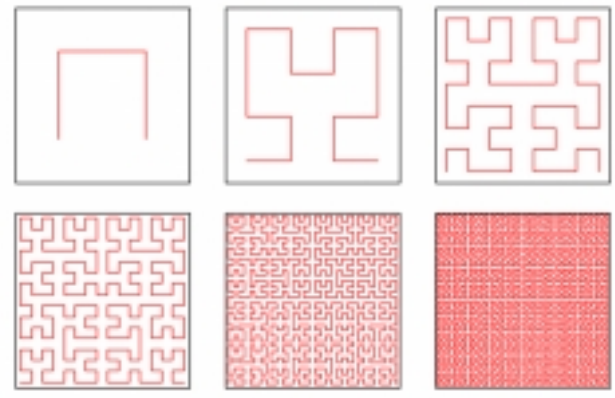

Figure 9. The Hilbert curve

References

[1] Mandelbrot BB. *The fractal geometry of nature*. Freeman, New York, 1983.

[2] Barnsley M F. *Fractals Everywhere*, 2nd ed, New York: Academic Press Professional; 1993.

[3] Lu.N. *Fractal imaging*. Academic Press , New York, 1997.

[4] H. O. Peitgen, H. Jurgens, and D. Saupe, *Chaos and Fractals: New Frontiers of Science*, New York, Springer-Verlag, Inc., 1992.

[5]. I. Stewart. 1988. A review of: the science of fractal images, *Nature* 336: 289.

[6] Falconer K. *Fractal geometry*: Mathematical foundations and applications., Second edition, Wiley, New York, 2003.