HOMOMORPHISMS IN THE ADEQUATE SEMIGROUPS OF THE TYPE A

Osman Hysa
"Aleksander Moisiu" University, Durrës, Mathematics Department
E-mail: **o.hysa@yahoo.com**

Abstract

Abundant semigroups have started being studied in the early '70s. They are a generalization of the class of regular semigroups, which is also their subclass. Later on two important subclasses were defined: quasi-adequate and adequate semigroups. Abundant semigroups are described by J. B. Fountain in his paper "Abundant Semigroups" (1979). He defined $*$ - relations P^*, Λ^*, D^*, J^* and H^* as a generalization of the known relations of Green P, Λ, D, J and H, in a given semigroup and also defined the abundant semigroups as semigroups in which every P^*- class and every Λ^*- class contain at least one idempotent. Quasi-adequate semigroups are abundant semigroups in the which the set of idempotents is a subsemigroup of them, while adequate semigroups are abundant semigroups in the which, for every two idempotents *e*, *f* from the set of their idempotents, we have *ef* = *fe*. Quasi-adequate and adequate semigroups are respectively subclasses of the known classes of semigroups: the orthodox and inverse semigroups. A particular importance in the study of algebraic structures has the study of the homomrphisms and isomorphisms between them, because they transfer many structural properties from an algebraic structure to another. A. El - Qallali and J. B. Fountain, in their paper "Quasi-adequate semigroups" (september 1981) have defined and studied the "good homomorphism" from the semigroup *S* in a semigroup *T*. They have also defined the "good congruence" in a given semigroup *S* and have shown some properties of these homomorphisms in the quasi-adequate semigroups. In this paper we will formulate and prove some analogous properties on the class of the adequate semigroups of the type A, that is a subclass of the adequate semigroups class.

***Keywords*:** abundant semigroup, adequate type A, good homomorphism, good congruence.

Introduction

Let be *S* a semigroup. In [1], J. B. Fountain, has defined the relations P^* and Λ^* in *S* as follows:

***Definition* 1:** *For the **e**lements a and b of the semigroup S we will say* $a\,P^*\,b$ $(a\,\Lambda^* b)$ *if we have* $a\,P\,b$ $(a\,\Lambda\,b)$ *in any oversemigroup of* S.

From this definition it follows immediately that the relations o * and Λ^* are respectively, left and right congruence in *S*.

***Definition* 2:** *A semigroup is called **abundant** if each* o **- class and each* Λ^**- class of this semigroup contains at least an idempotent*.

Definition* 3:** *The abundant semigroup* S *is called **quasi – adequate *semigroup, if the set of idempotents* $E(S)$ *is a subsemigroup of* S .

***Definition* 4:** *The quasi – adequate semigroup* S *is called **adequate** if the set of idempotents* $E(S)$ *is commutative subsemigroup of* S .

***Definition* 5:** *The adequate semigroup* S *is called adequate semigroup of the **type A**, if for every element* a *of* S *and for every idempotent* e *of* $E(S)$ *we have:*

$$eS \text{ Ç } aS = eaS \quad \text{and} \quad Se \text{ Ç } Sa = Sae$$

In [5], M. V. Lawson has defined the adequate semigroups of the type A as follows:

***Definition* 5’:** *The adequate semigroup S is called adequate of the **type A**, , if for every element a of S and for every idempotent e of $E(S)$ we have:*

$$ea = a(ea)^* \quad \textit{and} \quad ae = (ae)^+ a$$

We do not stop here to show the equivalence between the definitions 5 and 5 ', because it is not our purpose.

In the regular semigroups each o - class and each Λ- class contains at least an idempotent and also every semigroup includes itself. So, from the definition of the abundant semigroup, it follows immediatly that every regular semigroup is abundant. The orthodox semigroups and inverse semigroups are obviously abundants. Thus, the class of the regular semigroups is included in the class of the abundant semigroups.

In order to show $a \; \mathrm{P}^* b$, $a \; \Lambda^* b$, John Fountain in [1] has proved these equivalences:

$$a \; \mathrm{P}^* b \Leftrightarrow (\forall (x,y) \in S^1 \times S^1,\ xa = ya \vee xb = yb) \qquad (1)$$

$$a \; \Lambda^* b \Leftrightarrow (\forall (x,y) \in S^1 \times S^1,\ ax = ay \vee bx = by) \qquad (2)$$

He has also shown, as a corollary of (1) and (2), that if $e \in E(S)$ we have:

$$a \; \mathrm{P}^* e \Leftrightarrow ea = a \wedge (\forall (x,y) \in S^1 \times S^1,\ xa = ya \Rightarrow xe = ye) \qquad (3)$$

$$a \; \Lambda^* e \Leftrightarrow ae = a \wedge (\forall (x,y) \in S^1 \times S^1,\ ax = ay \Rightarrow ex = ey) \qquad (4)$$

The good homomorphism and the good congruence in a semigroup

A. El-Qallali and J. B. Fountain in [3], have studied the good homomorphism and the good congruence in a semigroup S. Previously, in [4], they have defined these two meanings as follows:

***Definition* 6:** *The homomorphism $f : S \to T$, where S and T are semigroups, is called **good homomorphism**, if for any two elements a,b of S we have*:

$$a \; \tilde{\mathrm{i}}(S)\; b \Rightarrow af \; \tilde{\mathrm{i}}(T)\; bf \quad \textit{and} \quad a \; \tilde{\mathrm{o}}(S)\; b \Rightarrow af \; \tilde{\mathrm{o}}(T)\; bf$$

***Definition* 7:** *The congruence ρ in a semigroup S is called **good congruence,** if the natural homomorphism $\psi : S \to S/\rho$ is a good homomorphism.*

In this paper we will formulate and prove some analogue propositions for the good homomorphisms and the good congruences in the adequate semigroups of the type A:

***Proposition* 1:** *If S is adequat semigroup of the type A and ρ is a good congruence in S, then for any idempotent $a\rho$ of the factor semigroup S/ρ will exist the idempotent $e \in E(S)$ such that $e\rho = a\rho$ (or $e\rho a$)*

***Proof*:** Let $a\rho$ be an idempotent in the factor semigroup S/ρ. Since S is adequat semigroup of the type A, i.e. is abundant, will exist the idempotents $e, f \in E(S)$ such that $f \; \tilde{\mathrm{o}}(S)\; a$ and $g \; \tilde{\mathrm{i}}(S)\; a$. But, on the other hand ρ is a good congruence, so from these two relations it follows:

$$(f\rho)\; \tilde{\mathrm{o}}(S/\rho)\; (a\rho) \text{ and } (g\rho)\; \tilde{\mathrm{i}}(S/\rho)\; (a\rho)$$

or

$$(f\rho)\; \mathrm{o}\; (a\rho) \text{ and } (g\rho)\; \mathrm{i}\; (a\rho)$$

in any semigroup T that includes the factor semigroup S/ρ as its subsemigroup.

We see also that the elements $(f\rho)$ and $(g\rho)$ are idempotents in (S/ρ). Indeed:

$$(f\rho)\cdot(f\rho) = [(f \cdot f)\rho] = (f\rho) \quad \text{and} \quad (g\rho)\cdot(g\rho) = [(g \cdot g)\rho] = (g\rho)$$

So we have:

$$(f\rho) = (a\rho)\cdot t \quad \wedge \quad (a\rho) = (f\rho)\cdot s \qquad (1)$$

and

$$(g\rho) = t'\cdot(a\rho) \quad \wedge \quad (a\rho) = s'\cdot(g\rho) \qquad (2)$$

where t, s, t', s' are elements of the semigroup *T*. From the equalities (1) and (2) it follows immediately:

$$(a\rho)\cdot(f\rho) = (f\rho) \quad \wedge \quad (f\rho)\cdot(a\rho) = (a\rho)$$

and

$$(a\rho)\cdot(g\rho) = (a\rho) \quad \wedge \quad (g\rho)\cdot(a\rho) = (g\rho)$$

Now, we indicate $e = f\cdot g$, that is an idempotent in $E(S)$, because S is aa adequate semigroup of the type A, that means $E(S)$ is a subsemigroup of S. For this idempotent we have:

$$\begin{aligned}(e\rho) &= ((f\cdot g)\rho) = (f)\rho\cdot(g)\rho = [(a\rho)(f)\rho]\cdot[(g\rho)(a)\rho] = \\ &= (a\rho)\cdot[(f)\rho\cdot(g\rho)]\cdot(a)\rho = (a\rho)\cdot[(f\cdot g)\rho]\cdot(a)\rho = \\ &= (a\rho)\cdot[(g\cdot f)\rho]\cdot(a)\rho = (a\rho)\cdot[(g)\rho\cdot(f\rho)]\cdot(a)\rho = \\ &= [(a\rho)(g)\rho]\cdot[(f\rho)(a)\rho] = (a\rho)\cdot(a\rho) = \\ &= (a\rho)\end{aligned}$$

So, we have finished the proof of this proposition. ■

***Proposition* 2:** *If S is an adequat semigroup of the type A and ρ is a good congruence in S, then the factor semigroup S/ρ is also an adequate semigroup of the type A.*

***Proof*:** First, we will show that the factor semigroup S/ρ is an abundant semigroup. Let $a\rho$ be an element of the factor semigroup S/ρ. Since S is adequate semigroup of the type A, it will be also abundant, so will exist the idempotent $e \in E(S)$ such that $e\,\mathcal{L}^{*}(S)\,a$. On the other hand ρ is a good congruence in S, that means the epiomorphism $S \to S/\rho$ is a good homomorphism, hence we will have:

$$(e\rho)\,\mathcal{L}^{*}(S/\rho)\,(a\rho) \quad \text{or} \quad (e\rho) \in R^{*}_{(a\rho)}$$

where $(e\rho) \in E(S/\rho)$. So, each $\mathcal{L}^{*}$- class in S/ρ has at least an idempotent, (it is the same to show that each $\mathcal{R}^{*}$- class has at least an idempotent), that means S/ρ is abundant.

Second, from the *proposition 1*, for each idempotent $a\rho \in S/\rho$ will exist the idempotent e in $E(S)$ such that $e \in a\rho$, i.e. $e\rho = a\rho$. So,

$$E(S/\rho) = \{e\rho : e \in E(S)\}$$

Thus, if $e\rho, f\rho \in E(S/\rho)$, (for $e, f \in E(S)$) we have: $(e\rho)\cdot(f\rho) = (ef)\rho$ and $(ef)\rho \in E(S/\rho)$ because $ef \in E(S)$. So, we have show that S/ρ is quasi-adequate semigroup. Moreover, since S is adequate semigroup of the type A, we have:

$$e\rho\cdot f\rho = (ef)\rho = (fe)\rho = f\rho\cdot e\rho$$

which means that S/ρ is adequate semigroup.

Finally, let $a\rho \in S/\rho$ and $e\rho \in E(S/\rho)$ and let's show that:

$$(e\rho)(S/\rho)\cap(a\rho)(S/\rho) = (e\rho)(a\rho)(S/\rho) \;\wedge\; (S/\rho)(e\rho)\cap(S/\rho)(a\rho) = (S/\rho)(a\rho)(e\rho)$$

Indeed,

$$x \in (e\rho)(S/\rho)\cap(a\rho)(S/\rho) \Rightarrow x = (e\rho)\cdot(s\rho)\wedge x = (a\rho)\cdot(t\rho) \Rightarrow x = (es)\rho \wedge x = (at)\rho$$

where $t, s \in S$. So, we have:

$$
\begin{aligned}
x = (ar)\cdot(tr) \Rightarrow\ & (er)\cdot x = (er)\cdot(ar)\cdot(sr) \Rightarrow \\
\Rightarrow\ & (er)\cdot[(es)r] = (er)\cdot(ar)\cdot(sr) \Rightarrow \\
\Rightarrow\ & (e\cdot es)r = (er)\cdot(ar)\cdot(sr) \Rightarrow \\
\Rightarrow\ & (es)r = (er)\cdot(ar)\cdot(sr) \Rightarrow \\
\Rightarrow\ & x = (er)\cdot(ar)\cdot(sr) \Rightarrow \\
\Rightarrow\ & x \in (er)\cdot(ar)\cdot(S/r)
\end{aligned}
$$

that means

$$(er)(S/r) \cap (ar)(S/r) \subseteq (er)(ar)(S/r) \qquad (3)$$

On the other hand we have:

$$x \in (er)(ar)(S/r) \Rightarrow x = (er)(ar)(sr) = (eas)r$$

for some $s \in S$. But, S is adequate semigroup of the type A, so:

$$eas \in eaS = eS \cap aS \Rightarrow eas \in aS \Rightarrow eas = as'$$

for some $s' \in S$. Then we have:

$$x = (er)\cdot(ar)\cdot(sr) = (er)\cdot(as)r \in (er)\cdot(S/r) \qquad (4)$$

$$x = (er)\cdot(ar)\cdot(sr) = (eas)r = (as')r = (ar)\cdot(s'r) \in (ar)\cdot(S/r) \qquad (4')$$

From (4) and (4') it follows:

$$(er)(S/r) \cap (ar)(S/r) \supseteq (er)(ar)(S/r) \qquad (5)$$

Now, from (3) and (5), derive:

$$(er)(S/r) \cap (ar)(S/r) = (er)(ar)(S/r)$$

In the same way we show that:

$$(S/r)(er) \cap (S/r)(ar) = (S/r)(ar)(er)$$

confirming the fact that, the factor semigroup (S/r) is adequate of the type A. ■

***Proposition* 3:** *Let $\psi : S \to T$ be a good homomorphism of the adequate semigroup of the type A, S in the semigroup T. If f is an idempotent in $S\psi$, then it will exist an idempotent e in S for which we will have $e\psi = f$ and moreover subsemigroup $S\psi$ of T is also adequate of the type A.*

***Proof*:** Let f be an idempotent in $S\psi$. That means there will exist an element a in S for which we have $a\psi = f$. Since S is adequate of the type A, the equivalence classes R_a^* and L_a^* containing respectively a single idempotent a^+ and a^*.Furthermore we see that the elements $a^+\psi$ and $a^*\psi$ are idempotents in the subsemigroup $S\psi$ of T:

$$a^+\psi \cdot a^+\psi = (a^+ \cdot a^+)\psi = a^+\psi \quad \text{and} \quad a^*\psi \cdot a^*\psi = (a^* \cdot a^*)\psi = a^*\psi$$

On the other hand, ψ is a good homomorphism so:

$$a^+ \mathcal{R}^*(S)\, a \Rightarrow a^+\psi\ \mathcal{R}^*(T)\, a\psi \Rightarrow a^+\psi\ \mathcal{R}(T')\, a\psi$$

and

$$a^* \mathcal{L}^*(S)\, a \Rightarrow a^*\psi\ \mathcal{L}^*(T)\, a\psi \Rightarrow a^*\psi\ \mathcal{L}(T'')\, a\psi$$

where T' and T'' are semigroups that include the semigroup T as their subsemigroup. From $a^+\psi\ \mathcal{R}(T')\, a\psi$ and $a^*\psi\ \mathcal{L}(T'')\, a\psi$ we obtain:

$$a\psi \cdot a^+\psi = a^+\psi;\quad a^+\psi \cdot a\psi = a\psi;\quad a^*\psi \cdot a\psi = a^*\psi;\quad a\psi \cdot a^*\psi = a\psi;$$

Now, taking $e = a^+a^*$, that is an idempotent in S as a production of two its idempotents, we have:

$$ey = (a^+a^*)y = a^+y \cdot a^*y = (ay \cdot a^+y)\cdot(a^*y \cdot ay) = ay \cdot (a^+y \cdot a^*y)\cdot ay =$$
$$= ay \cdot (a^+ \cdot a^*)y \cdot ay = ay \cdot (a^* \cdot a^+)y \cdot ay = ay \cdot (a^*y \cdot a^+y)\cdot ay =$$
$$= (ay \cdot a^*y)\cdot(a^+y \cdot ay) = (a \cdot a^*)y \cdot (a^+ \cdot a)y = ay \cdot ay = ay = f$$

Let's show now that the subsemigroup Sy of T is abundant. Let a be an element of Sy and x an element of S such that $xy = a$. Since S is adequate, the equivalence classes R_x^* and L_x^* according the relations $\mathcal{R}^*$ and $\mathcal{L}^*$ in S, containing respectively a single idempotent x^+ and x^*. So we have:

$$x^+ \mathcal{R}^*(S)\, x \Rightarrow x^+y\, \mathcal{R}^*(T)\, xy \Rightarrow x^+y\, \mathcal{R}(T')\, xy \Rightarrow x^+y\, \mathcal{R}^*(Sy)\, xy \Rightarrow x^+y\, \mathcal{R}^*(Sy)\, a$$

because $T' \supseteq T \supseteq Sy$. Hence, the equivalence class R_a^* of the $Sy / \mathcal{R}^*(Sy)$ contains the idempotent x^+y. In the same way we can show that the equivalence class L_a^* of the $Sy / \mathcal{L}^*(Sy)$ contains the idempotent x^*y. Thus, we have shown that the semigroup Sy is abundant.

Let's now show that the semigroup Sy is quasi-adequate. For this we must show that the set of idempotents $E(Sy)$ is a subsemigroup of Sy. So, let e, f be two idempotent of Sy. As shown in the first part of this proposition, will exist the idempotents e' and f' of the adequate semigroup S, such that $e'y = e$ and $f'y = f$. In these conditions we will have:

$$(ef)\cdot(ef) = (e'y \cdot f'y)\cdot(e'y \cdot f'y) = e'y \cdot (f'y \cdot e'y)\cdot f'y = e'y \cdot (f' \cdot e')y \cdot f'y =$$
$$= e'y \cdot (e' \cdot f')y \cdot f'y = e'y \cdot (e'y \cdot f'y)\cdot f'y = (e'y \cdot e'y)\cdot(f'y \cdot f'y) =$$
$$= (e' \cdot e')y \cdot (f' \cdot f')y = e'y \cdot f'y =$$
$$= ef$$

So Sy is quasi-adequate semigroup.

But, for the idempotents e, f of the semigroup Sy we have:

$$e \cdot f = e'y \cdot f'y = (e'f')y = (f'e')y = f'y \cdot e'y = f \cdot e$$

That means the Sy is adequate semigroup.

Now, we must show that, for each element a from Sy and for each idempotent e from $E(Sy)$ are true the equalities:

$$e \cdot Sy \cap a \cdot Sy = ea \cdot Sy \quad \wedge \quad Sy \cdot e \cap Sy \cdot a = Sy \cdot ae$$

If $x \in e \cdot Sy \cap a \cdot Sy$, then $x = e \cdot s_1 y \quad \wedge \quad x = a \cdot s_2 y$ where s_1 and s_2 are elements of S. From the first equality it follows that $ex = ee \cdot s_1 y = e \cdot s_1 y = x$ and by multiplying the second equality on the both sides by e, we will have :

$$x = ex = ea \cdot s_2 y \quad \Rightarrow \quad x \in ea \cdot Sy$$

that means:

$$e \cdot Sy \cap a \cdot Sy \subseteq ea \cdot Sy \qquad (6)$$

On the other hand, if $x \in ea \cdot Sy$ we have:

$$x \in e(a \cdot Sy) \Rightarrow x \in e \cdot Sy \qquad (7)$$

because $a \in Sy$. But, since $e \in Sy$ and $a \in Sy$, will exist $f \in E(S)$ and $b \in S$ such that $e \in fy \wedge a = by$. So,

$$x = (fy)\cdot(by)\cdot(sy) = (fbs)y \qquad (8)$$

Now, since $fbs \in fb \cdot S$ and S is adequate of the type A, it follows $f \cdot S \cap b \cdot S = fb \cdot S$, that means $fbs \in f \cdot S \wedge fbs \in b \cdot S$. Hence $fbs = b \cdot s'$, where $s' \in S$. Then,

$$x = (fbs)y = (bs')y = by \cdot s'y = a \cdot s'y \in a \cdot Sy \qquad (9)$$

From (7) and (9) we obtain: $x \in e \cdot Sy \cap a \cdot Sy$, that means

$$e \cdot Sy \cap a \cdot Sy \supset ea \cdot Sy \qquad (10)$$

Finally, from (6) and (10) we have:

$$e \cdot Sy \cap a \cdot Sy = ea \cdot Sy$$

So, the semigroup Sy is an adequate semigroup of the type A. ■

Reference

[1] John Fountain, Abundant Semigroups, *Proceedings of the Edinburgh Mathematical Society* (1979)

[2] John Fountain, Adequate Semigroups, *Proceedings of the Edinburgh Mathematical Society* (1979), 22, 113 – 125.

[3] A. El – Qallali, John Fountain, Quasi - adequate Semigroups, *Proceedings of the Royal Society of Edinburgh,* 91A, 91-99, 1981

[4] A. El Qallali and J. B. Fountain, Idempotent-connected abundant semigroups, Proc. Roy, Soc. Edinburgh, Sect. A 91 (1981), 79–90

[5] M. V. Lawson, Abundant rees matrix semigroups, *J. Austral. Math. Soc. (Series A)* 42 (1987), 132-142