BIFURCATION IN A DYNAMICAL SYSTEM: HARVEST MODELS

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Abstract

Most of the dynamical systems which are found in practice contain parameters which can be changed. It could happen that a slight change in the value of the parameter has a crucial influence in the behavior of the system. In particular, some fixed points can be destroyed, some new ones can be created, or the stability of some fixed points can change. These qualitative changes in dynamics are called bifurcations and the parameter values at which these occur are called bifurcation points. Bifurcation analysis has a lot of applications in different scientific fields, such as fluid mechanics, electronics, chemistry, ecology, medicine and economics. The objectives of this paper are to discuss some of the types of bifurcations in one-dimensional dynamical systems and application of these bifurcations in population dynamics, specifically in aquaculture. The majority of fish consumed by human comes from fishing in ocean and sea, but the natural supply is not sufficient to satisfy the increasing consumption of such resource and the cost of fish harvesting is increasing, so aquaculture has a great potential. Increasing aquaculture production will not only satisfy the demand of fish, but even will protect some high value species from extinction. Logistic growth model is used for population growth of fish, and three harvesting strategies are considered: constant, proportional and periodical harvesting. For each strategy is estimated the optimal amount of fish harvested to protect the population from extinction. The results indicate that harvesting in amount or rate higher than the bifurcation point brings the population to extinction. These findings can help fish farmers to ensure the continuation of fish population and to reduce the repopulation costs.

Keywords**:** *fixed point, stability, saddle- node, harvesting, logistic growth model.*

Introduction

Bifurcation theory is a key tool for the analysis of dynamical systems. This theory has become a major focus of researches in population dynamics during the last decades. Central to this topic is the question whether the qualitative properties of a dynamical system change when one or more of the parameters are changing. A bifurcation is a change in the number of fixed points or periodic orbits, or in their stability properties, as a parameter is varied. A bifurcation point is the value of parameter at which the change occurs. Bifurcations are classified according to how stability is lost. All types of bifurcations introduced are local bifurcations in the sense that only the behavior of a dynamical system in the neighborhood of a single fixed point is affected.

Population dynamics has attracted interest from the commercial harvesting industry and from many scientific communities including biology, ecology and economics. Mathematical models have been used widely to estimate the population dynamics of animals for so many years as well as the human population dynamics. In recent years, the use of mathematical models has been extended to agriculture sector especially in cattle farming to ensure continuous and optimum supply. The logistic growth model in term of harvesting has been used to study the fishery farming (Laham et al., 2012). Harvesting has been an area under discussion in population as well as in community dynamics (Murray 1993). The most important for successful management of harvested populations is that harvesting strategies are sustainable, not leading to instabilities or extinctions and produces great results for the year with little variation between the years (Aanes et al. 2002). Therefore, it can fulfill the market demand throughout the year.

The greater part of fish consumed by human comes from fishing in ocean/sea, but the natural supply is not yet sufficient to satisfy the increasing consumption, so aquaculture will have great potential all over the world, in the near future. In many countries the expansion of aquaculture is limited because of land and water availability, even if the demand for fish is increasing. Increasing aquaculture production will not only satisfy the global demand of fish, but even allow some stock of fish from the brink of extinction.

The use of mathematical models in fishery harvesting helps the aquaculture sector to estimate when and how many fish can be harvested to maximize the amount of fish obtained without completely depleting the population. This will enable them to be prepared with effective solutions to ensure that the fish's supplies can fulfill the consumer demand.

The objectives of this paper are: to discuss the saddle-node, transcritical and pitchfork bifurcations that occur in ordinary differential equations; to apply the bifurcation analysis to harvest models, using the logistic growth model and three harvesting strategies: constant harvesting, proportional harvesting and periodical harvesting; to estimate the optimum quantity of fish for harvesting that can ensure the continuous supply for each strategy and, to compare the results obtained between strategies.

In recent years, fish farming has been developed in Albania and aquaculture production constituted 17.6% of total fish caught in 2011, around 14% in 2010 from around 1% in 2001 (INSTAT database). Among Albanian products, processed fish products are successfully emerging in regional and European markets. Canned fish was the most exported product and constituted 25% of total agriculture products exported in 2010 (Ministry of Agriculture, Food and Consumer Protection, Albanian Agriculture 2011).

2. Bifurcation analysis in one dimension

Consider an ordinary differential equation of first order that depends on one parameter, r:

$$\dot{x} = f(x, r) \quad (1)$$

Let x^* be the equilibrium points of the equation (1). The bifurcations point ($x^*(r)$, r) are solutions of $f(x,r) = 0$ and $\frac{\partial f(x,r)}{\partial x} = 0$.

There are three common types of bifurcation: saddle node, pitchfork and transcritical [Strogatz, 2000; Teschl, 2004; Shone, 2002].

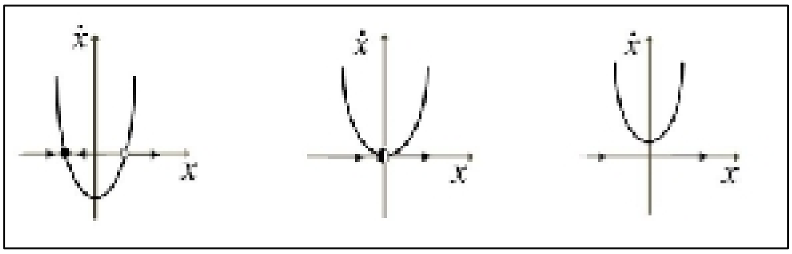
Saddle-node bifurcation

In saddle – node bifurcation, as the bifurcation parameter passes through the bifurcation point, two fixed points disappear, so there is no fixed point afterward. One of the two fixed points is stable and the other is unstable, before they disappear.

The normal form for a saddle-node bifurcation is given by

$$\dot{x} = r + x^2 \quad (2)$$

The fixed points are $x^* = \pm\sqrt{-r}$. It is clear that two real fixed points exist when $r < 0$. The stability of these fixed points when $r < 0$ are determined by the derivative of $f(x) = r + x^2$. Therefore, the negative fixed point is stable, and the positive fixed point is unstable (Figure 1a). When $r = 0$, the fixed points coalesce into a semi – stable fixed point at $x^* = 0$ (Figure 1b). This delicate fixed point, disappears when $r > 0$, and no real fixed point exist (Figure 1c). We say that a *bifurcation* occurred at $r = 0$, since the phase diagrams for $r < 0$ and $r > 0$ are qualitatively different.



a) $r < 0$ b) $r = 0$ c) $r > 0$

Figure 1

In dynamical systems, a bifurcation diagram shows the possible long-term values (fixed points or periodic orbits) of a system as a function of a bifurcation parameter in the system. It is usual to represent stable solutions with a solid line and unstable solutions with a dotted line. The bifurcation diagram for the saddle – node bifurcation indicate again that the two fixed points collide and annihilate at $r = 0$ and there are no fixed point for $r > 0$ (Figure 2).

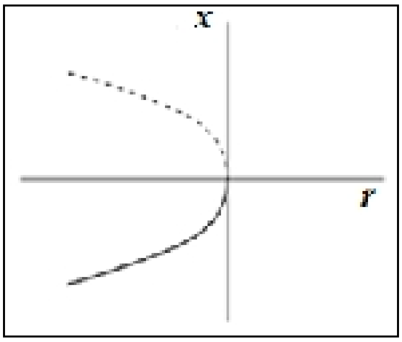


Figure 2

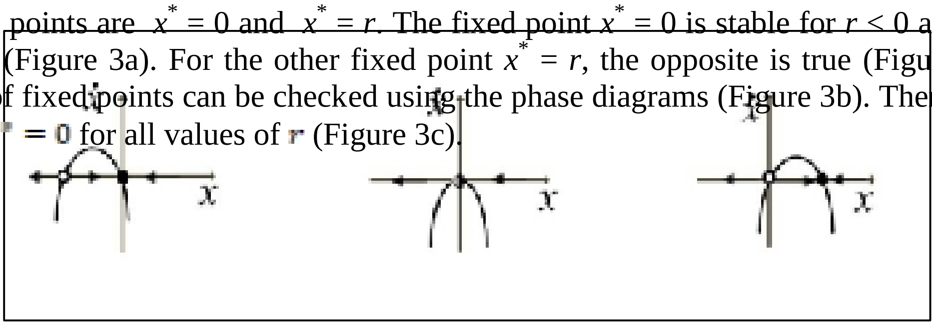
Transcritical bifurcation

There are certain scientific situations where a fixed point must exist for all values of a parameter and can never be destroyed. For example, in the logistic equation and other simple models for the growth of a single species, there is a fixed point at zero population, despite of the value of the growth rate. However, such a fixed point may change its stability as the parameter is varied. The transcritical bifurcation is the standard mechanism for such changes in stability.

The normal form for a transcritical bifurcation is given by

$$\dot{x} = rx - x^2 \qquad (3)$$

The fixed points are $x^* = 0$ and $x^* = r$. The fixed point $x^* = 0$ is stable for $r < 0$ and unstable for $r > 0$ (Figure 3a). For the other fixed point $x^* = r$, the opposite is true (Figure 3b). The stability of fixed points can be checked using the phase diagrams (Figure 3b). There is a fixed point at $x^* = 0$ for all values of r (Figure 3c).



a) $r < 0$ b) $r = 0$ c) $r > 0$

Figure 3

When the bifurcation point $r = 0$ is passed, there is an exchange of stability; the unstable fixed point becomes stable and the stable one becomes unstable. This is showed in the bifurcation diagram for the transcritical bifurcation (Figure 4).

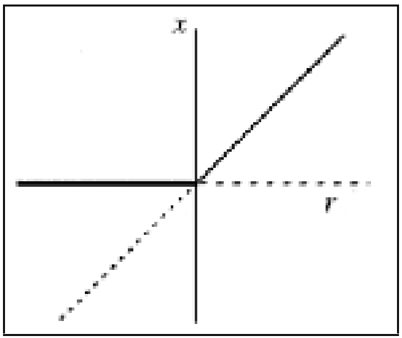


Figure 4

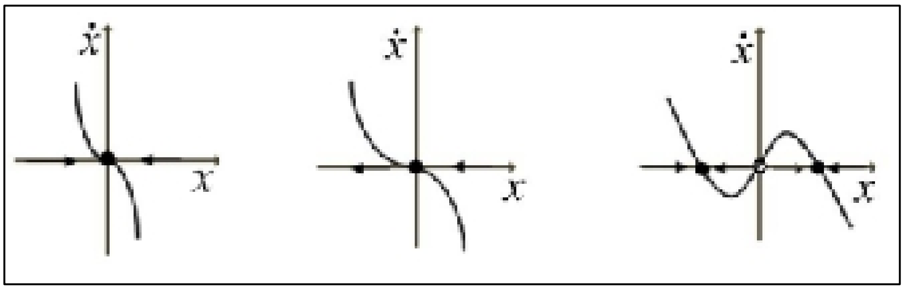
Supercritical Pitchfork Bifurcation

The pitchfork bifurcations occur in physical models where fixed points appear and disappear in pairs due to some certain symmetry of the problem. In the supercritical pitchfork bifurcation, a pair of stable fixed points is created at the bifurcation (or critical) point and exist after (super) the bifurcation. In the subcritical pitchfork bifurcation, a pair of unstable fixed points is created at the bifurcation point and exists before (sub) the bifurcation.

The normal form of the supercritical pitchfork bifurcation is given by

$$\dot{x} = rx - x^3 \qquad (4)$$

The fixed points are $x^* = 0$, and $x^* = \pm\sqrt{r}$. When $r < 0$, the fixed point $x^* = 0$ is the only fixed point, and it is stable (Figure 5a). When $r = 0$, the fixed point $x^* = 0$ is still stable (Figure 5b). When $r > 0$, the fixed point $x^* = 0$ is unstable and two other fixed points appear on either side of the origin, symmetrically located at $x^* = \pm\sqrt{r}$ which are stable (Figure 5b).



a) $r < 0$ b) $r = 0$ c) $r > 0$

Figure 5

Passing the bifurcation point $r = 0$, two nonzero stable fixed points are created after the stable zero fixed point, as is illustrated in the bifurcation diagram (Figure 6).

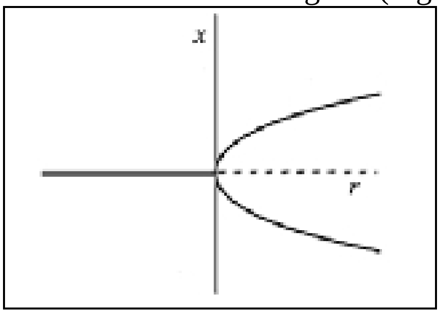


Figure 6

3. Application of bifurcations: fishery management mathematical models

We are going to analyze some simple fishery management models to illustrate the bifurcation analysis in a real world situation. We consider the logistic growth equation to model a fish population in the absence of fishing:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{M}\right) \qquad P(0) = P_0 \qquad (5)$$

where P is the size of the population, r is the growth rate due to reproduction and M is the carrying capacity of the environment.

Now consider modeling the population and harvesting some of the population using some common harvesting strategies: constant harvesting, proportional harvesting and periodic (seasonal) harvesting.

Constant harvesting is where a fixed numbers of fish were removed each year, while periodic harvesting is usually thought of a sequence of periodic closure and openings of different fishing grounds (Idels and Wang, 2008; Laham et al., 2012). In proportional harvesting, the quantity harvested is proportional to the population. Harvesting has been considered a factor of stabilization, destabilization, improvement of mean population levels, induced fluctuations, and control of non-native predators (Michel 2007).

We can use qualitative (geometric) analysis to estimate how many fish can be harvested and still allow the fish population to survive.

Constant harvesting

One of the simplest methods is the idea of harvesting where a set limit is established for harvesting. We assume that the dynamics of the population satisfies the logistic growth model (5) and that a constant harvesting, *h*, is added for removing a constant number of the fish over a given time interval. The mathematical model becomes:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{M}\right) - h, \quad P(0) = P_0 \tag{6}$$

The fixed points, P^*, are simply the solution of the equation $rP^*(1 - P^*/M) = h$. The model (6) has two fixed points $P^*_{1,2} = \frac{1}{2}\left(M \pm \sqrt{M^2 - \frac{4hM}{r}}\right)$ if $0 < h < rM/4$; one fixed point $P^* = M/2$ when $h = rM/4$; no fixed point when $h > rM/4$.

To illustrate the above results, consider a pond with a rare fish and rate of population growth $r = 0.6$, carrying capacity of the pond $M = 500$. The fixed point for model (5), where $h = 0$, are $P^* = 0$ and $P^* = M = 500$.

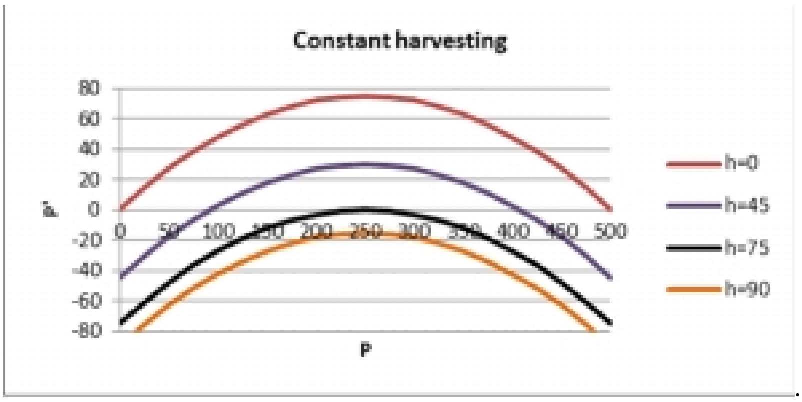
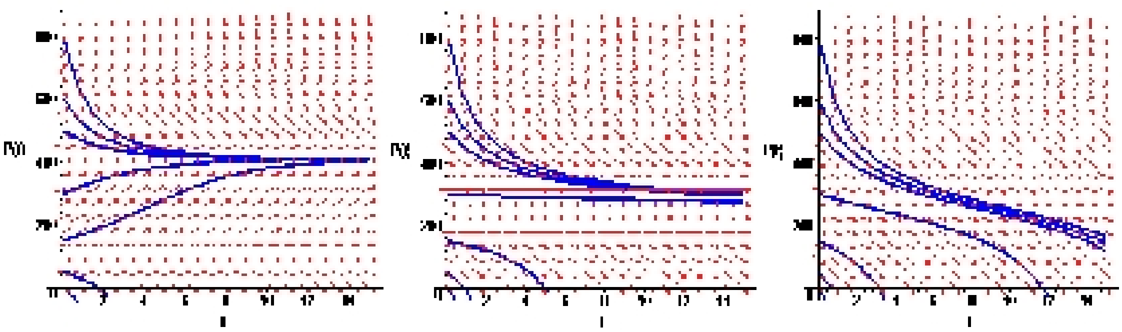


Figure 7

As the harvesting increases, the two fixed point move closer to each other with the lower fixed point remaining unstable and the upper fixed point remaining stable (Figure 7). As *h* move toward 75 (the maximum growth rate of the logistic growth equation), the two fixed points coalesce at $P^* = 250$. When $h > 75$, there is no fixed point, and the model shows that the population always goes extinct. This model shows a classic example of a *saddle node bifurcation*. The bifurcation values is $h = rM/4 = 75$.

The direction field of the differential equation for some values of h indicate first the existence of two fixed point (one stable and one unstable), then no fixed point. (Figure 8)



a)*h* = 45 b) *h* = 75 c) *h* = 90

Figure 8

The purpose of simple mathematical models applied to complex problems is to offer some insight. Here, the results indicate that overfishing (in the model $h > rM/4$) during one year can potentially result in a sudden collapse of the fish catch in subsequent years, so that governments need to be particularly cautious when contemplating increases in fishing quotas. With uncontrolled fish harvesting, a population could easily become extinct. That is why it is common to limit the amount of fish that can be harvested or to only allow fishing at certain times during the year.

Proportional harvesting

Another common form of harvesting is when one puts in a constant effort to harvest. In this case, the quantity harvested is proportional to the population. Thus, the mathematical model can be written:

$$\frac{dP}{dt} = rP\left(1-\frac{P}{M}\right) - hP, \quad P(0) = P_0 \qquad (7)$$

where again r is the growth rate, M is the carrying capacity with no harvesting and now h is the proportional rate of harvesting.

Algebric solution is complex and harder to interpret, thus we again turn to the geometric analysis of the model. The fixed points of (7) are the solution of the equation: $rP^*\left(1-\frac{P^*}{M}\right) = hP^*$, that is, $P^* = 0$ and $P^* = \frac{r-h}{rM}$. The extinction fixed point, $P^* = 0$, is unstable for values of $h < r$. As h increases, the larger equilibrium (carrying capacity) shrink, but it remains stable for $h < r$.

Consider the same data as the model with constant harvesting; only that now *h* is proportional rate of harvesting.

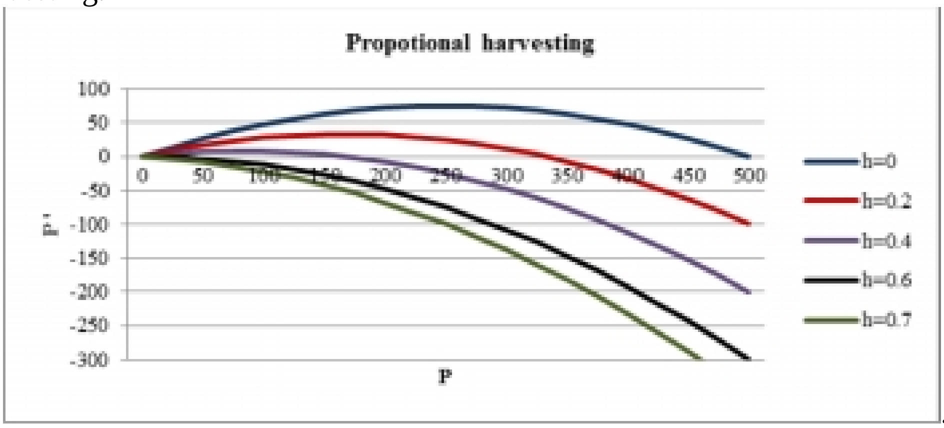


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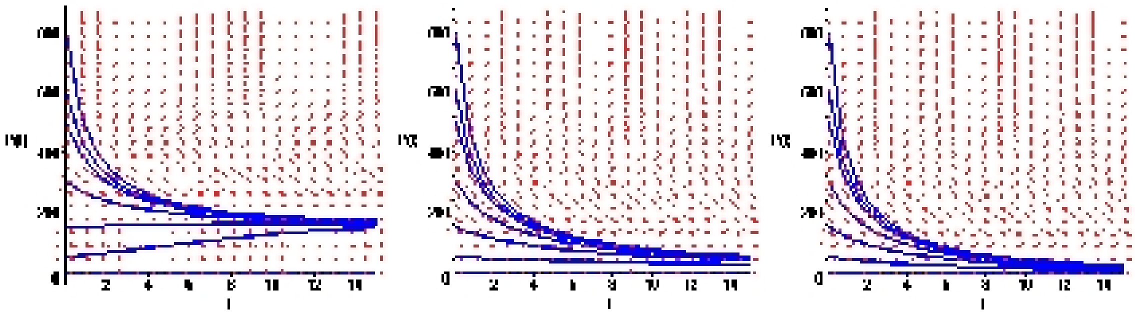
Figure 9

As the harvesting increases, the nontrivial fixed point move closer to extinction fixed point. As h move toward 0.6 (the growth rate), the nonzero fixed point fades to zero, which implies there is extinction because the harvesting rate approaches the growth rate. When $h > 0.6$, the rate of harvesting exceeds the reproduction rate and extinction necessarily follows. This model shows a classic example of a *transcritical bifurcation*. The bifurcation point is $h = 0.6$. The direction field of the differential equation for some values of h indicates the existence of two fixed point (one stable and one zero unstable), then one zero fixed point. (Figure 10).



a) $h = 0.4$ b) $h = 0.6$ c) $h = 0.7$

Figure 10

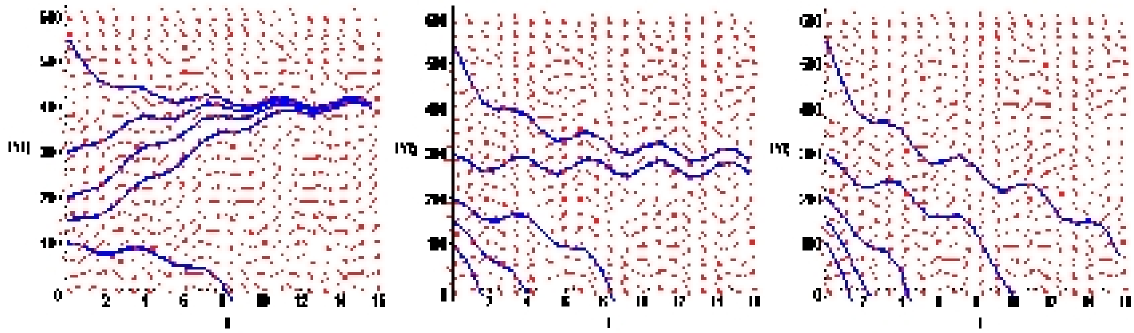
The results indicate that overfishing (in the model $h > r$) during one year can potentially extinct the fish in the pond, so again governments need to be particularly cautious when contemplating fishing quotas.

Periodic (seasonal) harvesting

Another very used form of harvesting is when harvesting is done during periods of time within a year, so the fish won't become extinct during fishing time and in some periods fishing is stopped, the population of fish might be able to increase again. The mathematical model can be written:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{M}\right) - h(1 + \sin 2\pi t) \qquad (8)$$

Consider the pond with growth rate $r = 0.6$ and carrying capacity $M = 500$, also an amount of fish is assumed for harvesting during a period of, for example, 6 months and followed by no harvesting for the next 6 months. This pattern repeats for several years. In order to ensure the population of fish is increasing, there is no harvesting in the next six months and the population of fish will increase until it approaches the carrying capacity.



a) $h = 45$ b) $h = 75$ c) $h = 90$

Figure 11

The model (8) is a non-autonomous differential equation, so the solutions are periodic and have the same general trend with the previous models. There are two solutions that oscillate about the fixed points (Figure 11.a). The solutions converge to one periodic solution that oscillates around the stable fixed point. For $h = 75$, there is only one fixed point (Figure 11.b). The population reaches the fixed point and stays there. As h increases more the population will extinct (Figure 11.c). Therefore, this periodic equation has the same bifurcation point as model (6) (Wood, 2009).

The periodic seasonal harvesting strategy optimizes the harvest while maintaining stable the population of fish. A harvesting strategy using logistic periodic seasonal harvesting strategy can be used to improve productivity, shorten investment return time and reduce risk from changes in sale price and costs of productions, particularly when comparatively short return periods are used (Laham et al., 2012).
It is common to have some months, for example 3 months, where heavy fishing is allowed and other months where only light fishing is allowed. The population still recovers to fixed point, but it takes longer for it to reach the stable fixed point because there are still a few fish that are being harvested during the rest of the year.

Conclusions

In practical applications that involve differential equations it very often happens that the differential equation contains parameters and the value of these parameters are often only known approximately. For that reason it is important to study the behavior of solutions and examine their dependence on the parameters. It can happen that a slight variation in a parameter can have significant impact on the solution.
We discussed three types of bifurcations: saddle-node bifurcations, supercritical pitchfork bifurcation, and transcritical bifurcation. In each case, the bifurcation value is at $r = 0$. These types of bifurcation also occur in higher-dimensional dynamical systems.
For successful management of harvested populations is very important that harvesting strategies are sustainable, not leading to instabilities or extinctions and produces great results for the year with little variation between the years. From the discussion of three harvesting strategies, results that using:

- the constant harvesting strategy, the fish population does not have enough time to recover if the constant harvesting is greater than the bifurcation point.
- the proportional harvesting strategy, the fish population will extinct if the proportional rate of harvesting is greater than the growth rate of the population or the bifurcation point.
- the periodic seasonal harvesting strategy optimizes the harvest while maintaining stable the population of fish, if the harvesting is lower or equal with the bifurcation point.

The development of appropriate fishery harvesting strategy can help the fulfillment of market demand. Supply of fish cannot rely only on the ocean/seas fishing activities, alternatives can be found by commercializing the aquaculture.
Other practical applications of bifurcations of one-dimensional or higher dimensional dynamical systems are the focus of future work.

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