

# HOMOMORPHISMS IN THE ADEQUATE SEMIGROUPS OF THE TYPE A

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## Abstract

Abundant semigroups have started being studied in the early '70s. They are a generalization of the class of regular semigroups, which is also their subclass. Later on two important subclasses were defined: quasi-adequate and adequate semigroups. Abundant semigroups are described by J. B. Fountain in his paper "Abundant Semigroups" (1979). He defined  $*$ -relations  $P^*$ ,  $\Lambda^*$ ,  $D^*$ ,  $J^*$  and  $H^*$  as a generalization of the known relations of Green  $P$ ,  $\Lambda$ ,  $D$ ,  $J$  and  $H$ , in a given semigroup and also defined the abundant semigroups as semigroups in which every  $P^*$ -class and every  $\Lambda^*$ -class contain at least one idempotent. Quasi-adequate semigroups are abundant semigroups in which the set of idempotents is a subsemigroup of them, while adequate semigroups are abundant semigroups in which, for every two idempotents  $e, f$  from the set of their idempotents, we have  $ef = fe$ . Quasi-adequate and adequate semigroups are respectively subclasses of the known classes of semigroups: the orthodox and inverse semigroups. A particular importance in the study of algebraic structures has the study of the homomorphisms and isomorphisms between them, because they transfer many structural properties from an algebraic structure to another. A. El - Qallali and J. B. Fountain, in their paper "Quasi-adequate semigroups" (september 1981) have defined and studied the "good homomorphism" from the semigroup  $S$  in a semigroup  $T$ . They have also defined the "good congruence" in a given semigroup  $S$  and have shown some properties of these homomorphisms in the quasi-adequate semigroups. In this paper we will formulate and prove some analogous properties on the class of the adequate semigroups of the type A, that is a subclass of the adequate semigroups class.

**Keywords:** abundant semigroup, adequate type A, good homomorphism, good congruence.

## Introduction

Let be  $S$  a semigroup. In [1], J. B. Fountain, has defined the relations  $P^*$  and  $\Lambda^*$  in  $S$  as follows:

**Definition 1:** For the elements  $a$  and  $b$  of the semigroup  $S$  we will say  $a P^* b$  ( $a \Lambda^* b$ ) if we have  $a P b$  ( $a \Lambda b$ ) in any oversemigroup of  $S$ .

From this definition it follows immediately that the relations  $C^*$  and  $\Lambda^*$  are respectively, left and right congruence in  $S$ .

**Definition 2:** A semigroup is called **abundant** if each  $C^*$ -class and each  $\Lambda^*$ -class of this semigroup contains at least an idempotent.

**Definition 3:** The abundant semigroup  $S$  is called **quasi – adequate** semigroup, if the set of idempotents  $E(S)$  is a subsemigroup of  $S$ .

**Definition 4:** The quasi – adequate semigroup  $S$  is called **adequate** if the set of idempotents  $E(S)$  is commutative subsemigroup of  $S$ .

**Definition 5:** The adequate semigroup  $S$  is called **adequate semigroup of the type A**, if for every element  $a$  of  $S$  and for every idempotent  $e$  of  $E(S)$  we have:

$$eS \subseteq aS \quad \text{and} \quad Se \subseteq Sa = Sae$$

In [5], M. V. Lawson has defined the adequate semigroups of the type A as follows:

**Definition 5':** The adequate semigroup  $S$  is called adequate of the type A, if for every element  $a$  of  $S$  and for every idempotent  $e$  of  $E(S)$  we have:

$$ea = a(ea)^* \quad \text{and} \quad ae = (ae)^+ a$$

We do not stop here to show the equivalence between the definitions 5 and 5', because it is not our purpose.

In the regular semigroups each  $\mathcal{C}$ -class and each  $\mathcal{L}$ -class contains at least an idempotent and also every semigroup includes itself. So, from the definition of the abundant semigroup, it follows immediately that every regular semigroup is abundant. The orthodox semigroups and inverse semigroups are obviously abundants. Thus, the class of the regular semigroups is included in the class of the abundant semigroups.

In order to show  $a \in P^*b, a \in \Lambda^*b$ , John Fountain in [1] has proved these equivalences:

$$a \in P^*b \Leftrightarrow (\exists (x,y) \in S^1 \times S^1, xa = ya \wedge xb = yb) \quad (1)$$

$$a \in \Lambda^*b \Leftrightarrow (\exists (x,y) \in S^1 \times S^1, ax = ay \wedge bx = by) \quad (2)$$

He has also shown, as a corollary of (1) and (2), that if  $e \in E(S)$  we have:

$$a \in P^*e \Leftrightarrow ea = a \wedge (\exists (x,y) \in S^1 \times S^1, xa = ya \wedge xe = ye) \quad (3)$$

$$a \in \Lambda^*e \Leftrightarrow ae = a \wedge (\exists (x,y) \in S^1 \times S^1, ax = ay \wedge ex = ey) \quad (4)$$

### The good homomorphism and the good congruence in a semigroup

A. El-Qallali and J. B. Fountain in [3], have studied the good homomorphism and the good congruence in a semigroup  $S$ . Previously, in [4], they have defined these two meanings as follows:

**Definition 6:** The homomorphism  $f : S \rightarrow T$ , where  $S$  and  $T$  are semigroups, is called **good homomorphism**, if for any two elements  $a, b$  of  $S$  we have:

$$a \in \mathcal{I}(S) b \Rightarrow af \in \mathcal{I}(T) bf \quad \text{and} \quad a \in \mathcal{C}(S) b \Rightarrow af \in \mathcal{C}(T) bf$$

**Definition 7:** The congruence  $r$  in a semigroup  $S$  is called **good congruence**, if the natural homomorphism  $\gamma : S \rightarrow S/r$  is a good homomorphism.

In this paper we will formulate and prove some analogue propositions for the good homomorphisms and the good congruences in the adequate semigroups of the type A:

**Proposition 1:** If  $S$  is adequate semigroup of the type A and  $r$  is a good congruence in  $S$ , then for any idempotent  $ar$  of the factor semigroup  $S/r$  will exist the idempotent  $e \in E(S)$  such that  $er = ar$  (or  $era$ )

**Proof:** Let  $ar$  be an idempotent in the factor semigroup  $S/r$ . Since  $S$  is adequate semigroup of the type A, i.e. is abundant, will exist the idempotents  $e, f \in E(S)$  such that  $f \in \mathcal{C}(S) a$  and  $g \in \mathcal{I}(S) a$ . But, on the other hand  $r$  is a good congruence, so from these two relations it follows:

$$(fr) \in \mathcal{C}(S/r) (ar) \quad \text{and} \quad (gr) \in \mathcal{I}(S/r) (ar)$$

or

$$(fr) \in \mathcal{C}(ar) \quad \text{and} \quad (gr) \in \mathcal{I}(ar)$$

in any semigroup  $T$  that includes the factor semigroup  $S/r$  as its subsemigroup.

We see also that the elements  $(fr)$  and  $(gr)$  are idempotents in  $(S/r)$ . Indeed:

$$(fr) \times (fr) = [(f \times f)r] = (fr) \quad \text{and} \quad (gr) \times (gr) = [(g \times g)r] = (gr)$$

So we have:

$$(fr) = (ar) \times \quad \wedge \quad (ar) = (fr) \times \quad (1)$$

and

$$(gr) \times (ar) = t' \times (ar) \quad \dot{\cup} \quad (ar) = s' \times (gr) \quad (2)$$

where  $t, s, t', s'$  are elements of the semigroup  $T$ . From the equalities (1) and (2) it follows immediately:

$$(ar) \times (fr) = (fr) \quad \dot{\cup} \quad (fr) \times (ar) = (ar)$$

and

$$1(ar) \times (gr) = (ar) \quad \dot{\cup} \quad (gr) \times (ar) = (gr)$$

Now, we indicate  $e = f \times g$ , that is an idempotent in  $E(S)$ , because  $S$  is an adequate semigroup of the type A, that means  $E(S)$  is a subsemigroup of  $S$ . For this idempotent we have:

$$\begin{aligned} (er) &= ((f \times g)r) = (f)r \times (g)r = [(ar)(f)r] \times [(gr)(a)r] = \\ &= (ar) \times [(f)r \times (gr)] \times (a)r = (ar) \times [(f \times g)r] \times (a)r = \\ &= (ar) \times [(g \times f)r] \times (a)r = (ar) \times [(g)r \times (f)r] \times (a)r = \\ &= [(ar)(g)r] \times [(f)r(a)r] = (ar) \times (ar) = \\ &= (ar) \end{aligned}$$

So, we have finished the proof of this proposition. ■

**Proposition 2:** If  $S$  is an adequate semigroup of the type A and  $r$  is a good congruence in  $S$ , then the factor semigroup  $S / r$  is also an adequate semigroup of the type A.

**Proof:** First, we will show that the factor semigroup  $S / r$  is an abundant semigroup. Let  $ar$  be an element of the factor semigroup  $S / r$ . Since  $S$  is an adequate semigroup of the type A, it will be also abundant, so will exist the idempotent  $e \in E(S)$  such that  $e \mathcal{C}^i(S) a$ . On the other hand  $r$  is a good congruence in  $S$ , that means the epimorphism  $S \rightarrow S / r$  is a good homomorphism, hence we will have:

$$(er) \mathcal{C}^i(S / r) (ar) \quad \text{or} \quad (er) \hat{\in} R_{(ar)}^*$$

where  $(er) \hat{\in} E(S / r)$ . So, each  $\mathcal{C}^i$ -class in  $S / r$  has at least an idempotent, (it is the same to show that each  $\mathcal{J}^*$ -class has at least an idempotent), that means  $S / r$  is abundant.

Second, from the proposition 1, for each idempotent  $ar \in S / r$  will exist the idempotent  $e \in E(S)$  such that  $e \mathcal{I} ar$ , i.e.  $er = ar$ . So,

$$E(S / r) = \{er : e \in E(S)\}$$

Thus, if  $er, fr \in E(S / r)$ , (for  $e, f \in E(S)$ ) we have:  $(er) \times (fr) = (ef)r$  and  $(ef)r \in E(S / r)$  because  $ef \in E(S)$ . So, we have shown that  $S / r$  is a quasi-adequate semigroup. Moreover, since  $S$  is an adequate semigroup of the type A, we have:

$$er \times fr = (ef)r = (fe)r = fr \times er$$

which means that  $S / r$  is an adequate semigroup.

Finally, let  $ar \in S / r$  and  $er \in E(S / r)$  and let's show that:

$$(er)(S / r) \mathcal{C} (ar)(S / r) = (er)(ar)(S / r) \quad \dot{\cup} \quad (S / r)(er) \mathcal{C} (S / r)(ar) = (S / r)(ar)(er)$$

Indeed,

$$x \hat{\in} (er)(S / r) \mathcal{C} (ar)(S / r) \quad \mathcal{P} \quad x = (er) \times (sr) \quad \dot{\cup} \quad x = (ar) \times (tr) \quad \mathcal{P} \quad x = (es)r \quad \dot{\cup} \quad x = (at)r$$

where  $t, s \in S$ . So, we have:



$$\begin{aligned}
ey &= (a^+ a^*)y = a^+ y \times a^* y = (ay \times a^+ y) \times (a^* y \times ay) = ay \times (a^+ y \times a^* y) \times ay = \\
&= ay \times (a^+ \times a^*)y \times ay = ay \times (a^* \times a^+)y \times ay = ay \times (a^* y \times a^+ y) \times ay = \\
&= (ay \times a^* y) \times (a^+ y \times ay) = (a \times a^*)y \times (a^+ \times a)y = ay \times ay = ay = f
\end{aligned}$$

Let's show now that the subsemigroup  $Sy$  of  $T$  is abundant. Let  $a$  be an element of  $Sy$  and  $x$  an element of  $S$  such that  $xy = a$ . Since  $S$  is adequate, the equivalence classes  $R_x^*$  and  $L_x^*$  according the relations  $C^*$  and  $\Lambda^*$  in  $S$ , containing respectively a single idempotent  $x^+$  and  $x^*$ . So we have:

$x^+ C^*(S) x \text{ P } x^+ y C^*(T) xy \text{ P } x^+ y C(T') xy \text{ P } x^+ y C^*(Sy) xy \text{ P } x^+ y C^*(Sy) a$   
because  $T' \hat{=} T \hat{=} Sy$ . Hence, the equivalence class  $R_a^*$  of the  $Sy / C^*(Sy)$  contains the idempotent  $x^+ y$ . In the same way we can show that the equivalence class  $L_a^*$  of the  $Sy / \Lambda^*(Sy)$  contains the idempotent  $x^* y$ . Thus, we have shown that the semigroup  $Sy$  is abundant.

Let's now show that the semigroup  $Sy$  is quasi-adequate. For this we must show that the set of idempotents  $E(Sy)$  is a subsemigroup of  $Sy$ . So, let  $e, f$  be two idempotent of  $Sy$ . As shown in the first part of this proposition, will exist the idempotents  $e'$  and  $f'$  of the adequate semigroup  $S$ , such that  $e'y = e$  and  $f'y = f$ . In these conditions we will have:

$$\begin{aligned}
(e'f) \times (ef) &= (e'y \times f'y) \times (e'y \times f'y) = e'y \times (f'y \times e'y) \times f'y = e'y \times (f' \times e')y \times f'y = \\
&= e'y \times (e' \times f')y \times f'y = e'y \times (e'y \times f'y) \times f'y = (e'y \times e'y) \times (f'y \times f'y) = \\
&= (e' \times e')y \times (f' \times f')y = e'y \times f'y = \\
&= ef
\end{aligned}$$

So  $Sy$  is quasi-adequate semigroup.

But, for the idempotents  $e, f$  of the semigroup  $Sy$  we have:

$$e \times f = e'y \times f'y = (e'f')y = (f'e')y = f'y \times e'y = f \times e$$

That means the  $Sy$  is adequate semigroup.

Now, we must show that, for each element  $a$  from  $Sy$  and for each idempotent  $e$  from  $E(Sy)$  are true the equalities:

$$e \times Sy \text{ Ç } a \times Sy = ea \times Sy \quad \dot{\cup} \quad Sy \times e \text{ Ç } Sy \times a = Sy \times ae$$

If  $x \hat{=} e \times Sy \text{ Ç } a \times Sy$ , then  $x = e \times_{s_1} y \quad \dot{\cup} \quad x = a \times_{s_2} y$  where  $s_1$  and  $s_2$  are elements of  $S$ . From the first equality it follows that  $ex = ee \times_{s_1} y = e \times_{s_1} y = x$  and by multiplying the second equality on the both sides by  $e$ , we will have :

$$x = ex = ea \times_{s_2} y \quad \text{P} \quad x \hat{=} ea \times Sy$$

that means:

$$e \times Sy \text{ Ç } a \times Sy \hat{=} ea \times Sy \quad (6)$$

On the other hand, if  $x \hat{=} ea \times Sy$  we have:

$$x \hat{=} e(a \times Sy) \text{ P } x \hat{=} e \times Sy \quad (7)$$

because  $a \hat{=} Sy$ . But, since  $e \hat{=} Sy$  and  $a \hat{=} Sy$ , will exist  $f \hat{=} E(S)$  and  $b \hat{=} S$  such that  $e \hat{=} fy \quad \dot{\cup} \quad a = by$ . So,

$$x = (fy) \times (by) \times (sy) = (fbs)y \quad (8)$$

Now, since  $fb \hat{=} fb \times S$  and  $S$  is adequate of the type A, it follows  $f \times S \zeta b \times S = fb \times S$ , that means  $fb \hat{=} f \times S \cup fb \hat{=} b \times S$ . Hence  $fb = b \times s'$ , where  $s' \hat{=} S$ . Then,

$$x = (fb)y = (bs')y = by \times s'y = a \times s'y \hat{=} a \times Sy \quad (9)$$

From (7) and (9) we obtain:  $x \hat{=} e \times Sy \zeta a \times Sy$ , that means

$$e \times Sy \zeta a \times Sy \hat{=} ea \times Sy \quad (10)$$

Finally, from (6) and (10) we have:

$$e \times Sy \zeta a \times Sy = ea \times Sy$$

So, the semigroup  $Sy$  is an adequate semigroup of the type A. ■

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