Spectral Properties and Tensor Products for quasi-class A_n^*

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Abstract:

In this article we will define the quasi-class A_n^* operator. An operator $T \in L(H)$, is said to belong to quasi-class A_n^* operator if

$$T^*\left(|T^{n+1}|^{\frac{2}{n+1}} - |T^*|^2\right)T \geq 0$$

for some positive integer n. We denote the set of $*-$class A_n by $Q(A_n^*)$. If T is A_n^*, then $T \in Q(A_n^*)$. Also, if n=1, then $Q(A_1^*)$ operators coincides with $Q(A^*)$ operators.

We will show basic structural properties and some spectral properties of this class of operators. We show that, if $T \in A_n^*$ then $\sigma_{jp}(T) = \sigma_p(T)$, $\sigma_{ja}(T)$, $\{0\} = \sigma_a(T)$, $\{0\}$, $T - \mu$ has finite ascent for all $\mu \in$, where it follows that T has SVEP. Also, we will prove here Browder's theorem, a-Browders theorem, and Tensor Products for quasi-class A_n^* operator.

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1. Introduction

Throughout this paper, let H be an infinite dimensional separable complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$. Let $L(H)$ denote the C^* algebra for all bounded operators on H. We shall denote the set of all complex numbers and the complex conjugate of a complex number λ by and $\bar{\lambda}$, respectively. The closure of a set M will be denoted by $\bar{M}$ and we shall henceforth shorten $T - \lambda I$ to $T - \lambda$. For $T \in L(H)$, we denote by $\ker T$ the null space and by $T(H)$ the range of T. We write $\alpha(T) = \dim\ker T$, $\beta(T) = \dim\ker T^* = \dim\overline{T(H)}^{\perp}$, and $\sigma(T)$ for the spectrum of T. We write $r(T)$ for the spectral radius. It is well known that

$r(T) \leq \| T \|$. The operator T is called normaloid, if $r(T) = \| T \|$. For an operator $T \in L(H)$, as usual, $|T| = (T^*T)^{\frac{1}{2}}$ and $[T^*, T] = T^*T - TT^*$ (the self-commutator of T). An operator $T \in L(H)$ is said to be normal, if $[T^*, T]$ is zero, and T is said te be hyponormal, if $[T^*, T]$ is nonnegative (equivalently if $|T| \geq |T^*|$). An operator $T \in L(H)$ is said to be paranormal [11], if $\| Tx \|^2 \leq \| T^2 x \|$ for any unit vector x in H. Further, T is said to be $*-$paranormal [4], if $\| T^* x \|^2 \leq \| T^2 x \|$ for any unit vector x in H. T is said to be k-paranormal operator if $\| Tx \|^{k+1} \leq \| T^{k+1} x \| \| x \|^k$ and T is said to be k-*-paranormal operator if $\| T^* x \|^{k+1} \leq \| T^{k+1} x \| \| x \|^k$. If T is k-*-paranormal, then T is (k+1)-paranormal operator. If T is k-paranormal operator, then T is normaloid [9].

T. Furuta, M. Ito and T. Yamazaki [12] introduced a very interesting class of bounded linear Hilbert space operators: class A defined by $|T^2| \geq |T|^2$, which is called the absolute value of T, and they showed that the class A is a subclass of paranormal operators. B. P. Dugall, I. H. Jeon, and I. H. Kim [10], introduced $*-$class A operator. An operator $T \in L(H)$ is said to be a $*-$class A operator, if $|T^2| \geq |T^*|^2$. A $*-$class A is a generalization of a hyponormal operator, [10, Theorem 1.2], and $*-$class A is a subclass of the class of $*-$paranormal operators, [10, Theorem 1.3]. We denote the set of $*-$class A by A^*. An operator $T \in L(H)$ is said to be a quasi$-*-$class A operator, if $T^* |T^2| T \geq T^* |T^*|^2 T$, [17]. We denote the set of quasi$-*-$class A by $Q(A^*)$. T. Furuta and J. Haketa [13], introduced n-perinormal operator: an operator $T \in L(H)$, is said to be n-perinormal operator, if $T^{*n}T^n \geq (T^*T)^n$, for each $n \geq 1$. An operator $T \in L(H)$, is said to be $n-*-$perinormal operator [6], if $T^{*n}T^n \geq (TT^*)^n$, for each $n \geq 1$. For $n = 1$, T is hyponormal operator, while, if T is 2-*-perinormal operator, then T is *-paranormal operator. If T is n-*-perinormal operator, then T is (n+1)-perinormal operator. In [20], Panayappan et al. define classes A_n operator: an operator $T \in L(H)$, is said to be A_n operator if $|T^{n+1}|^{\frac{2}{n+1}} \geq |T|^2$, for some positive integer n.

An operator $T \in L(H)$, is said to belongs to *-class A_n operator if

$$|T^{n+1}|^{\frac{2}{n+1}} \geq |T^*|^2$$

for some positive integer n.

We denote the set of $*-$class A_n by A_n^*. If n=1, then A_1^* coincides with the class A^* operator. If T is (n+1)-*-perinormal operator, then T is class A_n^*. If $T \in A_n^*$, then T is k-*-paranormal operator.

2. Spectral Properties of Quasi Class A_n^* Operator

Definition 2.1. An operator $T \in L(H)$, is said to belong to quasi-class A_n^* operator if

$$T^* \left(|T^{n+1}|^{\frac{2}{n+1}} - |T^*|^2 \right) T \geq 0$$

for some positive integer n.

We denote the set of $*-$class A_n by $Q(A_n^*)$.

If T is A_n^*, then $T \in Q(A_n^*)$. Also, if n=1, then $Q(A_1^*)$ operators coincides with $Q(A^*)$ operators.

Lemma 2.2. [14, Hansen Inequality] If $A, B \in L(H)$, satisfying $A \geq 0$ and $\| B \| \leq 1$, then

$$(B^* AB)^{¢} \geq B^* A^{¢} B \quad \text{for all} \quad ¢ \in (0,1].$$

Theorem 2.3. Let $T \in L(H)$ be a class $Q(A_n^*)$ operator for a positive integer n, T not have a dense range, and T let have the following representation

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \quad \text{on } H = \overline{T(H)} \oplus \ker T^*.$$

Then A is a class A_n^* on $\overline{T(H)}$, C= 0 and $\pm(T) = \pm(A) \cup \{0\}$.

Proof. Let P be the projection of H onto $\overline{T(H)}$, where $A = T|_{\overline{T(H)}}$ and $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} = TP = PTP.$ Since $T \in Q(A_n^*)$, we have

$$P\left(|T^{n+1}|^{\frac{2}{n+1}} - |T^*|^2 \right) P \geq 0.$$

We remark,

$$P|T^*|^2 P = PTT^*P = \begin{pmatrix} AA^* + BB^* & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} |A^*|^2 + |B^*|^2 & 0 \\ 0 & 0 \end{pmatrix}$$

and by Hansen Inequality, we have

$$P|T^{n+1}|^{\frac{2}{n+1}} P = P\left(T^{*(n+1)}T^{(n+1)}\right)^{\frac{1}{n+1}} P \leq \left(PT^{*(n+1)}T^{(n+1)}P\right)^{\frac{1}{n+1}} =$$

$$\left((TP)^{*(n+1)}(TP)^{(n+1)}\right)^{\frac{1}{n+1}} = \begin{pmatrix} |A^{n+1}|^2 & 0 \\ 0 & 0 \end{pmatrix}^{\frac{1}{n+1}} = \begin{pmatrix} |A^{n+1}|^{\frac{2}{n+1}} & 0 \\ 0 & 0 \end{pmatrix}$$

Then,

$$\begin{pmatrix} |A^{n+1}|^{\frac{2}{n+1}} & 0 \\ 0 & 0 \end{pmatrix} \geq P|T^{n+1}|^{\frac{2}{n+1}} P \geq P|T^*|^2 P = \begin{pmatrix} |A^*|^2 + |B^*|^2 & 0 \\ 0 & 0 \end{pmatrix} \geq \begin{pmatrix} |A^*|^2 & 0 \\ 0 & 0 \end{pmatrix},$$

so A is A_n^* operator on $\overline{T(H)}$.

Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in H = \overline{T(H)} \oplus \ker T^*$. Then,

$$\langle Cx_2, x_2 \rangle = \langle T(I-P)x, (I-P)y \rangle = \langle (I-P)x, T^*(I-P)y \rangle = 0,$$

thus $T^* = 0$.

By [15, Corollary 7], $\pm(A) \cup \pm(C) = \pm(T) \cup g$, where g is the union of the holes in $\pm(T)$, which happen to be a subset of $\pm(A) \cap \pm(C)$ and $\pm(A) \cap \pm(C)$ has no interior points. Therefore $\pm(T) = \pm(A) \cup \pm(C) = \pm(A) \cup \{0\}$.

The converse of the above theorem is valid.

Theorem 2.4. Let T be the operator on $\mathrm{H} = \overline{T(\mathrm{H})} \oplus \ker T^*$ defined as $T = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}$. If A belongs to class A_n^* operator, then T is quasi-class A_n^*.

Proof . We have

$$T^*\left(\left(T^{*(n+1)}T^{n+1}\right)^{\frac{1}{n+1}} - TT^*\right)T$$

$$= \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}^* \left\{\left(\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}^{*(n+1)} \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}^{n+1}\right)^{\frac{1}{n+1}} - \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}^*\right\}\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} A^*\left(\left(A^{*(n+1)}A^{n+1}\right)^{\frac{1}{n+1}} - AA^*\right)A & A^*\left(\left(A^{*(n+1)}A^{n+1}\right)^{\frac{1}{n+1}} - AA^*\right)B \\ B^*\left(\left(A^{*(n+1)}A^{n+1}\right)^{\frac{1}{n+1}} - AA^*\right)A & B^*\left(\left(A^{*(n+1)}A^{n+1}\right)^{\frac{1}{n+1}} - AA^*\right)B \end{pmatrix}$$

Let $v = x \oplus y$ be a vector in $\mathrm{H} = \overline{T(\mathrm{H})} \oplus \ker T^*$, where $x \in \overline{T(\mathrm{H})}$ and $y \in \ker T^*$. Then

$$\left\langle T^*\left((T^{*(n+1)}T^{n+1})^{\frac{1}{n+1}} - TT^*\right)Tv, v\right\rangle$$

$$= \left\langle A^*\left(\left|A^{n+1}\right|^{\frac{2}{n+1}} - |A^*|\right)Ax, x\right\rangle + \left\langle A^*\left(\left|A^{n+1}\right|^{\frac{2}{n+1}} - |A^*|\right)By, x\right\rangle$$

$$+ \left\langle B^*\left(\left|A^{n+1}\right|^{\frac{2}{n+1}} - |A^*|\right)Ax, y\right\rangle + \left\langle B^*\left(\left|A^{n+1}\right|^{\frac{2}{n+1}} - |A^*|\right)By, y\right\rangle$$

$$= \left\langle\left(\left|A^{n+1}\right|^{\frac{2}{n+1}} - |A^*|\right)(Ax + By), (Ax + By)\right\rangle \geq 0,$$

because A is A_n^* operator, $\left(\left|A^{n+1}\right|^{\frac{2}{n+1}} - |A^*|\right) \geq 0$. Hence T is quasi-class A_n^* operator.

Theorem 2.5. If $T \in \mathrm{Q}(\mathrm{A}_n^*)$ for a positive integer n and M be a closed T-invariant subspace, then the restriction $T|_{\mathrm{M}}$ is also $T \in \mathrm{Q}(\mathrm{A}_n^*)$ operator.

Proof. Let P be the projection of H onto M . Thus we can represent T as the following matrix with respect to the decomposition $M \oplus M^{\perp}$, $T = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$.

Put $A = T|_{\mathrm{M}}$ and we have $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} = TP = PTP$. Since $T \in \mathrm{Q}(\mathrm{A}_n^*)$, we have

$$PT^*\left(|T^{n+1}|^{\frac{2}{n+1}} - |T^*|^2\right)PT \geq 0.$$

We remark,

$$PT^*|T^*|^2\, TP = PT^*P|T^*|^2\, PTP = PT^*PTT^*PTP = \begin{pmatrix} A^*|A^*|^2\, A + |B^*A|^2 & 0 \\ 0 & 0 \end{pmatrix} \geq \begin{pmatrix} A^*|A^*|^2\, A & 0 \\ 0 & 0 \end{pmatrix}$$

and by Hansen inequality, we have

$$PT^{*}|T^{n+1}|^{\frac{2}{n+1}}TP = PT^{*}P\left(T^{*(n+1)}T^{(n+1)}\right)^{\frac{1}{n+1}}PTP \le PT^{*}\left(PT^{*(n+1)}T^{(n+1)}P\right)^{\frac{1}{n+1}}TP =$$

$$\begin{pmatrix} A^{*} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} |A^{n+1}|^{2} & 0 \\ 0 & 0 \end{pmatrix}^{\frac{1}{n+1}}\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A^{*} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} |A^{n+1}|^{\frac{2}{n+1}} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A^{*}|A^{n+1}|^{\frac{2}{n+1}}A & 0 \\ 0 & 0 \end{pmatrix}$$

Then,

$$\begin{pmatrix} A^{*}|A^{n+1}|^{\frac{2}{n+1}}A & 0 \\ 0 & 0 \end{pmatrix} \ge PT^{*}|T^{n+1}|^{\frac{2}{n+1}}TP \ge PT^{*}|T^{*}|^{2}TP \ge \begin{pmatrix} A^{*}|A^{*}|^{2}A & 0 \\ 0 & 0 \end{pmatrix}$$

so A is $Q(\mathcal{A}_n^{*})$ operator on $\mathcal{M}$.

A complex number λ is said to be in the point spectrum $\sigma_p(T)$ of T if there is a nonzero $x \in \mathcal{H}$ such that $(T-\lambda)x = 0$. If in addition, $(T-\lambda)^{*}x = 0$, then λ is said to be in the joint point spectrum $\sigma_{jp}(T)$ of T. Clearly $\sigma_{jp}(T) \subseteq \sigma_p(T)$. In general $\sigma_{jp}(T) \neq \sigma_p(T)$.

There are many classes of operators which

$$\sigma_{jp}(T) = \sigma_p(T) \qquad (1)$$

for example, if T is either normal or hyponormal operator. In [25] Xia showed that if T is a semihyponormal operator then holds (1). Dugall et.al extended this result to *-paranormal operators in [10]. In [17] the authors this result extended to quasi-class $\mathcal{A}^{*}$. Uchiyama, [24] showed that if T is class $\mathcal{A}$ operator then non zero points of $\sigma_{jp}(T)$ and $\sigma_p(T)$ are identical. The same thing is true for many operators' classes as well.

Here, we will tell that the equality (1) holds for class $T \in Q(\mathcal{A}_n^{*})$ operator.

Theorem 2.6. If $T \in Q(\mathcal{A}_n^{*})$ and $(T-\lambda)x = 0$, then $(T-\lambda)^{*}x = 0$ for all $\lambda \in \mathbb{C}$.

Proof . We may assume that $x \neq 0$. Let $\mathcal{M}$ be a span of $\{x\}$. Then $\mathcal{M}$ is an invariant subspace of T and let $T = \begin{pmatrix} \lambda & B \\ 0 & C \end{pmatrix}$ on $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$.

Let P be the projection of $\mathcal{H}$ onto $\mathcal{M}$, where $T|_{\mathcal{M}} = \lambda \neq 0$. For the proof, it is sufficient to show that B=0. Since $T \in Q(\mathcal{A}_n^{*})$ we have

$$P\left(|T^{n+1}|^{\frac{2}{n+1}} - |T^{*}|^{2}\right)P \ge 0.$$

By Hansen Inequality, we have

$$\begin{pmatrix} |\lambda|^{2} & 0 \\ 0 & 0 \end{pmatrix} = \left(PT^{*(n+1)}T^{(n+1)}P\right)^{\frac{1}{n+1}} \ge P\left(T^{*(n+1)}T^{(n+1)}\right)^{\frac{1}{n+1}}P = P|T^{n+1}|^{\frac{2}{n+1}}P \ge P|T^{*}|^{2}P = \begin{pmatrix} |\lambda|^{2}+|B^{*}|^{2} & 0 \\ 0 & 0 \end{pmatrix}$$

Thus, B=0.

Corollary 2.7. If $T \in Q(\mathcal{A}_n^{*})$, then $\sigma_{jp}(T) = \sigma_p(T)$.

Corollary 2.8. If $T \in Q(\mathcal{A}_n^{*})$, then $(T-\lambda) \le \mu(T-\lambda)$ for all $\lambda \in \mathbb{C}$.

Proof. It is obvious from Theorem 2.6.

Theorem 2.9. If $T \in Q(\mathcal{A}_n^{*})$ and $\mu, \nu \in \sigma_p(T)$ with $\mu \neq \nu$, then $\ker(T-\mu) \perp \ker(T-\nu)$.

Proof. Let $x \in \ker(T-\mu)$ and $y \in \ker(T-\nu)$. Then $Tx = \mu x$ and $Ty = \nu y$. Therefore

$$\mu\langle x, y\rangle = \langle Tx, y\rangle = \langle x, T^{*}y\rangle = \bar{\nu}\langle x, y\rangle,$$

then $\langle x, y\rangle = 0$. Therefore, $\ker(T-\mu) \perp \ker(T-\nu)$.

Theorem 2.10. If $T \in Q(A_n^*)$ has the representation $T = \lambda \oplus A$ on $\ker(T-\lambda) \oplus \ker(T-\lambda)^{\perp}$, where λ is an eigenvalue of T, then A belongs to class $Q(A_n^*)$, with $\ker(A-\lambda)=\{0\}$.

Proof. Since $T = \lambda \oplus A$, then $T = \begin{pmatrix} \lambda & 0 \\ 0 & A \end{pmatrix}$ and we have

$$T^* |T^{n+1}|^{\frac{2}{n+1}} T - T^* |T^*|^2 T =$$

$$\begin{pmatrix} \lambda^4 & 0 \\ 0 & A^* |A^{n+1}|^{\frac{2}{n+1}} A \end{pmatrix} - \begin{pmatrix} \lambda^4 & 0 \\ 0 & A^* |A^*|^2 A \end{pmatrix} = \begin{pmatrix} \lambda^4 & 0 \\ 0 & A^* |A^{n+1}|^{\frac{2}{n+1}} A - A^* |A^*|^2 A \end{pmatrix}$$

Since $T \in Q(A_n^*)$, then $A \in Q(A_n^*)$.

Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \ker(A-\lambda)$, then

$$(T-\lambda)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & A-\lambda \end{pmatrix}\begin{pmatrix} 0 \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

so $x \in \ker(T-\lambda)$, therefore x=0.

Lemma 2.11. [5, Holder-McCarthy inequality] Let T be a psositive operator. Then the following inequalities hold for all $x \in H$

1). $\langle T^r x, x\rangle \le \langle Tx, x\rangle^r \|x\|^{2(1-r)}$ for $0 < r < 1$,

2) $\langle T^r x, x\rangle \ge \langle Tx, x\rangle^r \|x\|^{2(1-r)}$ for $r \ge 1$.

A complex number λ is said to be in the approximate point spectrum $\sigma_a(T)$ of T if there is a sequence $\{x_n\}$ of unit vectors satisfying $(T-\lambda)x_n \to 0$. If in additions $(T-\lambda)^* x_n \to 0$ then λ is said to be in the joint approximate point spectrum $\sigma_{ja}(T)$ of operator T. Clearly $\sigma_{ja}(T) \subseteq \sigma_a(T)$. In general $\sigma_{ja}(T) \neq \sigma_a(T)$.

There are many classes of operators for which

$$\sigma_{ja}(T) = \sigma_a(T) \qquad (2)$$

for example, if T is either normal or hyponormal operator. In [25] Xia showed that if T is a semihyponormal operator then holds (2). Dugall et.al extended this result to *-paranormal operators in [10]. Cho and Yamazaki in [7] showed that if T is class A operator, then nonzero points of $\sigma_{ja}(T)$ and $\sigma_a(T)$ are identical. In the following, we will show that if $T \in Q(A_n^*)$, then nonzero points of $\sigma_{ja}(T)$ and $\sigma_a(T)$ are identical.

Theorem 2.12. If T is of the class $Q(A_n^*)$ operator, and $(T-\lambda)x_m \to 0$ for $\lambda \neq 0$, then $(T-\lambda)^* x_m \to 0$.

Proof. Let T be a class $Q(A_n^*)$ operator and $\|(T-\lambda)x_m\| \to 0$. We may assume that $\|x_m\| = 1$. By the assumption $(T-\lambda)x_m \to 0$, from

$$T^k = (T-\lambda+\lambda)^k = \sum_{i=1}^{k} \binom{k}{i} \lambda^{k-i} (T-\lambda)^i + \lambda^k, \quad \text{for } k \in \ ,$$

we have $(T^{2+n} - \lambda^{2+n})x_m \to 0$.

By

$$\left| \| T^{2+n}x_m \| - |\lambda|^{2+n} \right| \leq \| (T^{2+n} - \lambda^{2+n})x_m \|$$

hence

$$\| T^{2+n}x_m \| \to |\lambda|^{2+n} \qquad (3)$$

Moreover

$$\left| \| T^{*}\lambda x_m \| - \| T^{*}(T-\lambda)x_m \| \right| \leq \| T^{*}Tx_m \| . \qquad (4)$$

So

Since $T \in Q(A_n^*)$, by Holder-McCarthy inequality, we get

$$\| T^{*}Tx_m \|^2 = \left\langle T^{*}|T^{*}|^2\, Tx_m, x_m \right\rangle \leq \left\langle T^{*}|T^{1+n}|^{\frac{2}{1+n}}\, Tx_m, x_m \right\rangle \leq$$

$$\left\langle |T^{1+n}|^2\, Tx_m, Tx_m \right\rangle^{\frac{1}{1+n}} \|Tx\|^{\frac{2n}{n+1}} = \| T^{2+n}x_m \|^{\frac{2}{1+n}} \| Tx \|^{\frac{2n}{n+1}}$$

So,

$$\| T^{*}Tx_m \| \leq \| T^{2+n}x_m \|^{\frac{1}{1+n}} \| Tx \|^{\frac{n}{n+1}}$$

Then it follows from (3), (4) and (5) that

$$\limsup_{m\to\infty} \| T^{*}x_m \| \leq |\lambda| .$$

Since

$$\left\|(T-\lambda)^{*}x_m\right\|^2 = \langle T^{*}x_m, T^{*}x_m\rangle - \bar{\lambda}\langle x_m, T^{*}x_m\rangle - \lambda\langle T^{*}x_m, x_m\rangle + |\lambda|^2 =$$

$$\| T^{*}x_m \|^2 - \bar{\lambda}\langle Tx_m, x_m\rangle - \lambda\langle x_m, Tx_m\rangle + |\lambda|^2 = \| T^{*}x_m \|^2 - \bar{\lambda}\left\langle (T-\lambda)x_m, x_m \right\rangle - \lambda\left\langle x_m, (T-\lambda)x_m \right\rangle - |\lambda|^2$$

then

$$\limsup_{m\to\infty} \| (T-\lambda)^{*}x_m \|^2 \leq |\lambda|^2 - |\lambda|^2 = 0 .$$

This implies $(T-\lambda)^{*}x_m \to 0.$

Corollary 2.13. If T is of the class $Q(A_n^*)$ operator, then $\sigma_{ja}(T) \setminus \{0\} = \sigma_a(T) \setminus \{0\}$.

Lemma 2.14.[3] Let $T = U|T|$ be the polar decomposition of T, $\lambda \neq 0$ and $\{x_m\}$ a sequence of vectors. Then the following assertions are equivalent:

1). $(T-\lambda)x_m \to 0$ and $(T^{*}-\bar{\lambda})x_m \to 0,$

2). $(|T| - |\lambda|)x_m \to 0$ and $(U - e^{i\theta})x_m \to 0,$

3). $(|T^{*}| - |\lambda|)x_m \to 0$ and $(U^{*} - e^{-i\theta})x_m \to 0.$

Theorem 2.15. If T is of the class $Q(A_n^*)$ operator and $\lambda \in \sigma_a(T) \setminus \{0\}$ then $|\lambda| \in \sigma_a(|T|) \cap \sigma_a(|T^{*}|).$

Proof. If $\lambda \in \sigma_a(T) \setminus \{0\}$, then by Theorem 2.12., there exists a sequence of unit vectors $\{x_m\}$ such that $(T-\lambda)x_m \to 0$ and $(T-\lambda)^{*}x_m \to 0$. Hence, from Lemma 2.14. we have $|\lambda| \in \sigma_a(|T|) \cap \sigma_a(|T^{*}|).$

Let $Hol(\sigma(T))$ be the space of all analytic functions in an open neighborhood of $\sigma(T)$. We say that $T \in L(H)$ has the single valued extension property at $\lambda \in$, if for every open neighborhood U of λ the only analytic function $f: U \to \mathbb{C}$ which satisfies equation $(T-\lambda)f(\lambda)=0,$ is the constant function f\equiv 0. The operator T is said to have SVEP if T has SVEP at every $\lambda \in \mathbb{C}$. An operator $T \in L(H)$ has SVEP at every point of the resolvent $\rho(T) = \mathbb{C} \setminus \sigma(T).$ Every operator T has SVEP at an isolated point of the spectrum.

For $T \in L(H),$ the smallest nonnegative integer p such that $\ker T^p = \ker T^{p+1}$ is called the ascent of T and is denoted by p(T). If no such integer exists, we set $p(T) = \infty.$ We say that $T \in L(H)$ is of finite ascent (finitely ascensive) if $p(T-\lambda) < \infty,$ for all $\lambda \in \mathbb{C}$..

Corollary 2.16. If T is of the class $Q(A_n^*)$ operator, then $T-\lambda$ has finite ascent for all $\lambda \in \mathbb{C}$.

Proof. We have to tell that $\ker(T-\lambda) = \ker(T-\lambda)^2$. To do that, it is sufficient enough to show that $\ker(T-\lambda)^2 \subseteq \ker(T-\lambda)$, since $\ker(T-\lambda) \subseteq \ker(T-\lambda)^2$ is clear.

Let $x \in \ker(T-\lambda)^2$, then $(T-\lambda)^2 x = 0$. From Theorem 2.6. we have $(T-\lambda)^*(T-\lambda)x = 0$. Hence,

$$\|(T-\lambda)x\|^2 = \langle (T-\lambda)^*(T-\lambda)x, x \rangle = 0,$$

so we have $(T-\lambda)x = 0$, which implies $\ker(T-\lambda)^2 \subseteq \ker(T-\lambda)$.

Corollary 2.17. If T is of the class $Q(A_n^*)$ operator, then T has SVEP.

Proof. Proof, obvious from [2, Theorem 2.39].

An operator $T \in L(H)$ is called an upper semi-Fredholm, if it has a closed range and $\alpha(T) < \infty$, while T is called a lower semi-Fredholm if $\beta(T) < \infty$. However, T is called a semi-Fredholm operator if T is either an upper or a lower semi-Fredholm, and T is said to be a Fredholm operator if it is both an upper and a lower semi-Fredholm. If $T \in L(H)$ is semi-Fredholm, then the index is defined by

$$\mathrm{ind}(T) = \alpha(T) - \beta(T).$$

An operator $T \in L(H)$ is said to be upper semi-Weyl operator if it is upper semi-Fredholm and $\mathrm{ind}(T) \le 0,$ while T is said to be lower semi-Weyl operator if it is lower semi-Fredholm and $\mathrm{ind}(T) \ge 0.$ An operator is said to be Weyl operator if it is Fredholm of index zero.

The Weyl spectrum and the essential approximate spectrum are defined by

$$\sigma_w(T) = \{\lambda \in \mathbb{C} : T-\lambda \text{ is not Weyl}\},$$

and

$$\sigma_{uw}(T) = \{\lambda \in \mathbb{C} : T-\lambda \text{ is not upper semi-Weyl}\}.$$

An operator $T \in L(H)$ is said to be upper semi-Browder operator, if it is upper semi-Fredholm and $p(T) < \infty$. An operator $T \in L(H)$ is said to be lower semi-Browder operator, if it is lower semi-Fredholm and $q(T) < \infty$. An operator $T \in L(H)$ is said to be Browder operator, if it is Fredholm of finite ascent and descent. The Browder spectrum and the upper semi-Browder spectrum are defined by

$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T-\lambda \text{ is not Browder}\},$$

and

$$\sigma_{ub}(T) = \{\lambda \in \mathbb{C} : T-\lambda \text{ is not upper semi-Browder}\}.$$

Theorem 2.18. If T or T^* belongs to class $Q(A_n^*),$ then $\sigma_w(f(T)) = f(\sigma_w(T))$ for all $f \in Hol(\sigma(T)).$

Proof. The inclusion $f(\pm_w(T)) \subseteq \pm_w(f(T))$ holds for any operator. Let $T \in Q(A_n^*)$, then T has SVEP, then from [2, Theorem 4.19] holds $\pm_w(f(T)) \subseteq f(\pm_w(T))$. If $T^* \in Q(A_n^*)$, similar to above.

The following concept has been introduced in 1997 by Harte and W.Y. Lee [16]: A bounded operator T is said to satisfy Browder's theorem if $\pm_w(T) = \pm_b(T)$.

Theorem 2.19. If T or T^* belongs to class $Q(A_n^*)$, then f(T) satisfy Browder's theorem for all $f \in Hol(\pm(T))$.

Proof. Since T or T^* has SVEP, then from [2, Theorem 4.22] f(T) satisfies Browder's theorem for all $f \in Hol(\pm(T))$.

The following concept has been introduced in 2000, [8]: A bounded operator T is said to satisfy a-Browder's theorem if $\pm_{uw}(T) = \pm_{ub}(T)$.

Theorem 2.20. If T or T^* belongs to class $Q(A_n^*)$, then a-Browder's theorem holds for f(T) and $f(T)^*$ for all $f \in Hol(\pm(T))$.

Proof. Since T or T^* has SVEP, then from [2,Theorem 4.33] f(T) and $f(T)^*$ satisfies Browder's theorem for all $f \in Hol(\pm(T))$.

3. Tensor products for quasi *-class A_n

Let H and K denote the Hilbert spaces. For given non zero operators $T \in L(H)$ and $S \in L(K)$, $T \otimes S$ denotes the tensor product on the product space $H \otimes K$. The normaloid property is invariant under tensor products, [22]. There exist paranormal operators T and S, such that $T \otimes S$ is not paranormal, [1]. In [23], Stochel proved $T \otimes S$ is normal, if and only if, T and S are normal. This result was extended to class A operators, *-class A operators, class A_n operators and quasi class A_n operators in [18], [10], [20], and [21] respectively. In this section, we prove an analogues result for $Q(A_n^*)$ operators.

Lemma 3.1. [9] Let $T \in L(H)$ and $S \in L(K)$ be non zero operators. Then:

1). $(T \otimes S)^*(T \otimes S) = T^*T \otimes S^*S$,

2). $|T \otimes S|^p = |T|^p \otimes |S|^p$ for any positive real number p.

Theorem 3.2. Let $T \in L(H)$ and $S \in L(K)$ be non zero operators. Then $T \otimes S$ belongs to $Q(A_n^*)$ operator, if and only if, one of the following holds:

1). T and S are $Q(A_n^*)$,

2). $T^2 = 0$ or $S^2 = 0$.

Proof. We have

$$(T \otimes S)^* \left(|(T \otimes S)^{n+1}|^{\frac{2}{n+1}} - |(T \otimes S)^*|^2 \right)(T \otimes S) =$$

$$(T \otimes S)^* \left(|T^{n+1}|^{\frac{2}{n+1}} \otimes |S^{n+1}|^{\frac{2}{n+1}} - |T^*|^2 \otimes |S^*|^2 \right)(T \otimes S) =$$

$$T^* \left(|T^{n+1}|^{\frac{2}{n+1}} - |T^*|^2 \right) \otimes S^* |S^{n+1}|^{\frac{2}{n+1}} S + T^* |T^*|^2 T \otimes S^* \left(|S^{n+1}|^{\frac{2}{n+1}} - |S^*|^2 \right) S$$

Hence, if either (1) T and S are $Q(A_n^*)$ or (2) $T^2 = 0$ or $S^2 = 0$, then $T \otimes S \in Q(A_n^*)$.

Conversely, suppose that $T \otimes S$ is a $Q(A_n^*)$ operator. Then, for $x \in H$, $y \in K$ we gain

$$\left\langle T^*\left(|T^{n+1}|^{\frac{2}{n+1}} - |T^*|^2\right)Tx, x\right\rangle\left\langle S^*|S^{n+1}|^{\frac{2}{n+1}} Sy, y\right\rangle + \left\langle T^*|T^*|^2 Tx, x\right\rangle\left\langle (S^*|S^{n+1}|^{\frac{2}{n+1}} - |S^*|^2)Sy, y\right\rangle \geq 0$$

It suffices to show that if the statement (1) does not hold, then the statement (2) holds. Thus, assume to the contrary that neither T^2 nor S^2 is the zero operator, and T is not $Q(A_n^*)$ operator. Then, there exists $x_0 \in H$, such that:

$$\left\langle T^*\left(|T^{n+1}|^{\frac{2}{n+1}} - |T^*|^2\right)Tx_0, x_0\right\rangle = \sim < 0 \text{ and } \left\langle T^*|T^*|^2 Tx_0, x_0\right\rangle = \quad > 0.$$

From above relation, we have

$$(\sim + \quad)\left\langle S^*|S^{n+1}|^{\frac{2}{n+1}} Sy, y\right\rangle \geq \quad \left\langle S^*|S^*|^2 Sy, y\right\rangle.$$

Thus, $S \in Q(A^*)$ operator, because $\sim + \quad < \quad$.

We have,

$$\left\langle S^*|S^*|^2 Sy, y\right\rangle = \left\langle |S^*|^2 Sy, Sy\right\rangle = \langle S^*Sy, S^*Sy\rangle = \| S^*Sy \|^2$$

and using the Holder McCarthy inequality, we get

$$\left\langle S^*|S^{n+1}|^{\frac{2}{n+1}} Sy, y\right\rangle = \left\langle \left(S^{*(n+1)}S^{n+1}\right)^{\frac{1}{n+1}} Sy, Sy\right\rangle \leq \left\langle S^{*(n+1)}S^{n+1}Sy, Sy\right\rangle^{\frac{1}{n+1}} \| Sy \|^{\frac{2n}{n+1}} = \| S^{n+2}y \|^{\frac{2}{n+1}} \| Sy \|^{\frac{2n}{n+1}}$$

Then ,

$$(\sim + \quad) \| S^{n+2}y \|^{\frac{2}{n+1}} \| Sy \|^{\frac{2n}{n+1}} \geq \quad \| S^*Sy \|^2 .$$

Since $S \in Q(A_n^*)$, from Theorem 2.3. S has decomposition of the form

$$S = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \text{ on } H = \overline{S(H)} \oplus kerS^*$$

where $A = S|_{\overline{S(H)}}$ is A_n^* operator, we have

$$(\sim + \quad) \| A^{n+2}« \|^{\frac{2}{n+1}} \| A« \|^{\frac{2n}{n+1}} \geq \quad \| A^*A« \|^2,$$

for all $« \in \overline{S(H)}$.

Since $A \in A_n^*$, then A is normaloid, since A is k-*-paranormal operator. Thus, taking supremum on both sides of the above inequality, we have

$$(\sim + \quad) \| A \|^4 \geq \quad \| A \|^4 .$$

This inequality makes A=0. Hence, $S^2 = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}^2 = 0.$ This is a contradiction to that S^2 is not a zero operator. So T must be a $Q(A_n^*)$ operator. A similar argument shows that S is also a $Q(A_n^*)$ operator, which completes the proof.

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