

Spectral Properties and Tensor Products for quasi-class A_n^*

Ilmi Hoxha and Naim L. Braha

Department of Mathematics and Computer Sciences,
Avenue "Mother Theresa " 5, Prishtine, 10000, Kosova
E-mail address:ilmihoxha011@hotmail.com, nbraha@yahoo.com.

Abstract:

In this article we will define the quasi-class A_n^* operator. An operator $T \in L(H)$, is said to belong to quasi-class A_n^* operator if

$$T^* \left(|T^{n+1}|^{\frac{2}{n+1}} - |T^*|^2 \right) T \geq 0$$

for some positive integer n . We denote the set of $*$ -class A_n by $Q(A_n^*)$. If T is A_n^* , then $T \in Q(A_n^*)$. Also, if $n=1$, then $Q(A_1^*)$ operators coincides with $Q(A^*)$ operators.

We will show basic structural properties and some spectral properties of this class of operators. We show that, if $T \in A_n^*$ then $\uparrow_{jp}(T) = \uparrow_p(T)$, $\uparrow_{ja}(T)$, $\{0\} = \uparrow_a(T)$, $\{0\}$, $T - \}$ has finite ascent for all $\} \in$, where it follows that T has SVEP. Also, we will prove here Browder's theorem, a -Browders theorem, and Tensor Products for quasi-class A_n^* operator.

AMS Mathematics Subject Classification (2000): 47B20, 47B37.

Key words and phrases: quasi-class A_n^* operator, SVEP, Ascent, Browder's theorem, a – Browders theorem, Tensor Products.

1. Introduction

Throughout this paper, let H be an infinite dimensional separable complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Let $L(H)$ denote the C^* algebra for all bounded operators on H . We shall denote the set of all complex numbers and the complex conjugate of a complex number \sim by $\bar{\sim}$ and \sim^{-} , respectively. The closure of a set M will be denoted by \bar{M} and we shall henceforth shorten $T \sim I$ to $T \sim$. For $T \in L(H)$, we denote by $\ker T$ the null space and by $T(H)$ the range of T . We write $r(T) = \dim \ker T$, $s(T) = \dim \ker T^* = \dim \overline{T(H)}^\perp$, and $\sigma(T)$ for the spectrum of T . We write $r(T)$ for the spectral radius. It is well known that $r(T) \leq \|T\|$. The operator T is called normaloid, if $r(T) = \|T\|$. For an operator $T \in L(H)$, as usual, $|T| = (T^*T)^{\frac{1}{2}}$ and $[T^*, T] = T^*T - TT^*$ (the self-commutator of T). An operator $T \in L(H)$ is said to be normal, if $[T^*, T]$ is zero, and T is said to be hyponormal, if $[T^*, T]$ is nonnegative (equivalently if $|T| \geq |T^*|$). An operator $T \in L(H)$ is said to be paranormal [11], if $\|Tx\|^2 \leq \|T^2x\|$ for any unit vector x in H . Further, T is said to be $*$ -paranormal [4], if $\|T^*x\|^2 \leq \|T^2x\|$ for any unit vector x in H . T is said to be k -paranormal operator if $\|Tx\|^{k+1} \leq \|T^{k+1}x\| \|x\|^k$ and T is said to be k - $*$ -paranormal operator if $\|T^*x\|^{k+1} \leq \|T^{k+1}x\| \|x\|^k$. If T is k - $*$ -paranormal, then T is $(k+1)$ -paranormal operator. If T is k -paranormal operator, then T is normaloid [9].

T . Furuta, M. Ito and T. Yamazaki [12] introduced a very interesting class of bounded linear Hilbert space operators: class A defined by $|T^2| \geq |T|^2$, which is called the absolute value of T , and they showed that the class A is a subclass of paranormal operators. B. P. Dugall, I. H. Jeon, and I. H. Kim [10], introduced $*$ -class A operator. An operator $T \in L(H)$ is said to be a $*$ -class A operator, if $|T^2| \geq |T^*|^2$. A $*$ -class A is a generalization of a hyponormal operator, [10, Theorem 1.2], and $*$ -class A is a subclass of the class of $*$ -paranormal operators, [10, Theorem 1.3]. We denote the set of $*$ -class A by A^* . An operator $T \in L(H)$ is said to be a quasi- $*$ -class A operator, if $T^*|T^2|T \geq T^*|T^*|^2T$, [17]. We denote the set of quasi- $*$ -class A by $Q(A^*)$. T . Furuta and J. Haketa [13], introduced n -perinormal operator: an operator $T \in L(H)$, is said to be n -perinormal operator, if $T^{*n}T^n \geq (T^*T)^n$, for each $n \geq 1$. An operator $T \in L(H)$, is said to be n - $*$ -perinormal operator [6], if $T^{*n}T^n \geq (TT^*)^n$, for each $n \geq 1$. For $n = 1$, T is hyponormal operator, while, if T is 2- $*$ -perinormal operator, then T is $*$ -paranormal operator. If T is n - $*$ -perinormal operator, then T is $(n+1)$ -perinormal operator. In [20], Panayappan et al. define classes A_n operator: an operator $T \in L(H)$, is said to be A_n operator if $|T^{n+1}|^{\frac{2}{n+1}} \geq |T|^2$, for some positive integer n .

An operator $T \in L(H)$, is said to belongs to $*$ -class A_n operator if

$$|T^{n+1}|^{\frac{2}{n+1}} \geq |T^*|^2$$

for some positive integer n .

We denote the set of $*$ -class A_n by A_n^* . If $n=1$, then A_1^* coincides with the class A^* operator. If T is $(n+1)$ - $*$ -perinormal operator, then T is class A_n^* . If $T \in A_n^*$, then T is k - $*$ -paranormal operator.

2. Spectral Properties of Quasi Class A_n^* Operator

Definition 2.1. An operator $T \in L(H)$, is said to belong to quasi-class A_n^* operator if

$$T^* \left(|T^{n+1}|^{\frac{2}{n+1}} - |T^*|^2 \right) T \geq 0$$

for some positive integer n .

We denote the set of $*$ -class A_n by $Q(A_n^*)$.

If T is A_n^* , then $T \in Q(A_n^*)$. Also, if $n=1$, then $Q(A_1^*)$ operators coincides with $Q(A^*)$ operators.

Lemma 2.2. [14, Hansen Inequality] If $A, B \in L(H)$, satisfying $A \geq 0$ and $\|B\| \leq 1$, then

$$(B^*AB)^u \geq B^*A^uB \quad \text{for all } u \in (0,1].$$

Theorem 2.3. Let $T \in L(H)$ be a class $Q(A_n^*)$ operator for a positive integer n , T not have a dense range, and T let have the following representation

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \quad \text{on } H = \overline{T(H)} \oplus \ker T^*.$$

Then A is a class A_n^* on $\overline{T(H)}$, $C=0$ and $\dagger(T) = \dagger(A) \cup \{0\}$.

Proof. Let P be the projection of H onto $\overline{T(H)}$, where $A = T|_{\overline{T(H)}}$ and

$$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} = TP = PTP. \quad \text{Since } T \in Q(A_n^*), \text{ we have}$$

$$P \left(|T^{n+1}|^{\frac{2}{n+1}} - |T^*|^2 \right) P \geq 0.$$

We remark,

$$P |T^*|^2 P = PTT^*P = \begin{pmatrix} AA^* + BB^* & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} |A^*|^2 + |B^*|^2 & 0 \\ 0 & 0 \end{pmatrix}$$

and by Hansen Inequality, we have

$$\begin{aligned} P |T^{n+1}|^{\frac{2}{n+1}} P &= P \left(T^{*(n+1)} T^{(n+1)} \right)^{\frac{1}{n+1}} P \leq \left(PT^{*(n+1)} T^{(n+1)} P \right)^{\frac{1}{n+1}} = \\ & \left((TP)^{*(n+1)} (TP)^{(n+1)} \right)^{\frac{1}{n+1}} = \begin{pmatrix} |A^{n+1}|^2 & 0 \\ 0 & 0 \end{pmatrix}^{\frac{1}{n+1}} = \begin{pmatrix} |A^{n+1}|^{\frac{2}{n+1}} & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

Then,

$$\begin{pmatrix} |A^{n+1}|^{\frac{2}{n+1}} & 0 \\ 0 & 0 \end{pmatrix} \geq P |T^{n+1}|^{\frac{2}{n+1}} P \geq P |T^*|^2 P = \begin{pmatrix} |A^*|^2 + |B^*|^2 & 0 \\ 0 & 0 \end{pmatrix} \geq \begin{pmatrix} |A^*|^2 & 0 \\ 0 & 0 \end{pmatrix},$$

so A is A_n^* operator on $\overline{T(H)}$.

Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in H = \overline{T(H)} \oplus \ker T^*$. Then,

$$\langle Cx_2, x_2 \rangle = \langle T(I - P)x, (I - P)y \rangle = \langle (I - P)x, T^*(I - P)y \rangle = 0,$$

thus $T^* = 0$.

By [15, Corollary 7], $\dagger(A) \cup \dagger(C) = \dagger(T) \cup [$, where $[$ is the union of the holes in $\dagger(T)$, which happen to be a subset of $\dagger(A) \cap \dagger(C)$ and $\dagger(A) \cap \dagger(C)$ has no interior points. Therefore $\dagger(T) = \dagger(A) \cup \dagger(C) = \dagger(A) \cup \{0\}$.

The converse of the above theorem is valid.

Theorem 2.4. Let T be the operator on $H = \overline{T(H)} \oplus \ker T^*$ defined as $T = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}$. If A

belongs to class A_n^* operator, then T is quasi-class A_n^* .

Proof . We have

$$\begin{aligned} & T^* \left(\left(T^{*(n+1)} T^{n+1} \right)^{\frac{1}{n+1}} - T T^* \right) T \\ &= \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}^* \left\{ \left(\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}^{*(n+1)} \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}^{n+1} \right)^{\frac{1}{n+1}} - \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}^* \right\} \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} A^* \left(\left(A^{*(n+1)} A^{n+1} \right)^{\frac{1}{n+1}} - A A^* \right) A & A^* \left(\left(A^{*(n+1)} A^{n+1} \right)^{\frac{1}{n+1}} - A A^* \right) B \\ B^* \left(\left(A^{*(n+1)} A^{n+1} \right)^{\frac{1}{n+1}} - A A^* \right) A & B^* \left(\left(A^{*(n+1)} A^{n+1} \right)^{\frac{1}{n+1}} - A A^* \right) B \end{pmatrix} \end{aligned}$$

Let $v = x \oplus y$ be a vector in $H = \overline{T(H)} \oplus \ker T^*$, where $x \in \overline{T(H)}$ and $y \in \ker T^*$.

Then

$$\begin{aligned} & \left\langle T^* \left(\left(T^{*(n+1)} T^{n+1} \right)^{\frac{1}{n+1}} - T T^* \right) T v, v \right\rangle \\ &= \left\langle A^* \left(\left| A^{n+1} \right|^{\frac{2}{n+1}} - |A^*| \right) A x, x \right\rangle + \left\langle A^* \left(\left| A^{n+1} \right|^{\frac{2}{n+1}} - |A^*| \right) B y, x \right\rangle \\ &+ \left\langle B^* \left(\left| A^{n+1} \right|^{\frac{2}{n+1}} - |A^*| \right) A x, y \right\rangle + \left\langle B^* \left(\left| A^{n+1} \right|^{\frac{2}{n+1}} - |A^*| \right) B y, y \right\rangle \\ &= \left\langle \left(\left| A^{n+1} \right|^{\frac{2}{n+1}} - |A^*| \right) (A x + B y), (A x + B y) \right\rangle \geq 0, \end{aligned}$$

because A is A_n^* operator, $\left(\left| A^{n+1} \right|^{\frac{2}{n+1}} - |A^*| \right) \geq 0$. Hence T is quasi-class A_n^* operator.

Theorem 2.5. If $T \in Q(A_n^*)$ for a positive integer n and M be a closed T-invariant subspace, then the restriction $T|_M$ is also $T \in Q(A_n^*)$ operator.

Proof. Let P be the projection of H onto M . Thus we can represent T as the following matrix with respect to the decomposition $M \oplus M^\perp$, $T = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$.

Put $A = T|_M$ and we have $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} = TP = PTP$. Since $T \in Q(A_n^*)$, we have

$$PT^* \left(|T^{n+1}|^{\frac{2}{n+1}} - |T^*|^2 \right) PT \geq 0.$$

We remark,

$$PT^* |T^*|^2 TP = PT^* P |T^*|^2 PTP = PT^* PTT^* PTP = \begin{pmatrix} A^* |A^*|^2 A + |B^* A|^2 & 0 \\ 0 & 0 \end{pmatrix} \geq \begin{pmatrix} A^* |A^*|^2 A & 0 \\ 0 & 0 \end{pmatrix}$$

and by Hansen inequality, we have

$$PT^* |T^{n+1}|^{\frac{2}{n+1}} TP = PT^* P \left(T^{*(n+1)} T^{(n+1)} \right)^{\frac{1}{n+1}} PTP \leq PT^* \left(PT^{*(n+1)} T^{(n+1)} P \right)^{\frac{1}{n+1}} TP =$$

$$\begin{pmatrix} A^* & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} |A^{n+1}|^2 & 0 \\ 0 & 0 \end{pmatrix}^{\frac{1}{n+1}} \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A^* & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} |A^{n+1}|^{\frac{2}{n+1}} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A^* |A^{n+1}|^{\frac{2}{n+1}} A & 0 \\ 0 & 0 \end{pmatrix}$$

Then,

$$\begin{pmatrix} A^* |A^{n+1}|^{\frac{2}{n+1}} A & 0 \\ 0 & 0 \end{pmatrix} \geq PT^* |T^{n+1}|^{\frac{2}{n+1}} TP \geq PT^* |T^*|^2 TP \geq \begin{pmatrix} A^* |A^*|^2 A & 0 \\ 0 & 0 \end{pmatrix}$$

so A is $Q(A_n^*)$ operator on M .

A complex number λ is said to be in the point spectrum $\sigma_p(T)$ of T if there is a nonzero $x \in H$ such that $(T - \lambda)x = 0$. If in addition, $(T - \lambda)^* x = 0$, then λ is said to be in the joint point spectrum $\sigma_{jp}(T)$ of T . Clearly $\sigma_{jp}(T) \subseteq \sigma_p(T)$. In general $\sigma_{jp}(T) \neq \sigma_p(T)$.

There are many classes of operators which

$$\sigma_{jp}(T) = \sigma_p(T) \quad (1)$$

for example, if T is either normal or hyponormal operator. In [25] Xia showed that if T is a semihyponormal operator then holds (1). Dugall et.al extended this result to $*$ -paranormal operators in [10]. In [17] the authors this result extended to quasi-class A^* . Uchiyama, [24] showed that if T is class A operator then non zero points of $\sigma_{jp}(T)$ and $\sigma_p(T)$ are identical. The same thing is true for many operators' classes as well.

Here, we will tell that the equality (1) holds for class $T \in Q(A_n^*)$ operator.

Theorem 2.6. If $T \in Q(A_n^*)$ and $(T - \lambda)x = 0$, then $(T - \lambda)^* x = 0$ for all $\lambda \in \dots$

Proof . We may assume that $x \neq 0$. Let M be a span of $\{x\}$. Then M is an invariant subspace of T and let $T = \begin{pmatrix} \lambda & B \\ 0 & C \end{pmatrix}$ on $H = M \oplus M^\perp$.

Let P be the projection of H onto M , where $T|_M = \lambda \neq 0$. For the proof, it is sufficient to show that $B=0$. Since $T \in Q(A_n^*)$ we have

$$P \left(|T^{n+1}|^{\frac{2}{n+1}} - |T^*|^2 \right) P \geq 0.$$

By Hansen Inequality, we have

$$\begin{pmatrix} |\lambda|^2 & 0 \\ 0 & 0 \end{pmatrix} = (PT^{*(n+1)}T^{(n+1)}P)^{\frac{1}{n+1}} \geq P(T^{*(n+1)}T^{(n+1)})^{\frac{1}{n+1}}P = P|T^{n+1}|^{\frac{2}{n+1}}P \geq P|T^*|^2P = \begin{pmatrix} |\lambda|^2 + |B^*|^2 & 0 \\ 0 & 0 \end{pmatrix}$$

Thus, $B=0$.

Corollary 2.7. If $T \in Q(A_n^*)$, then $\dagger_{jp}(T) = \dagger_p(T)$.

Corollary 2.8. If $T \in Q(A_n^*)$, then $s(T - \lambda) \leq r(T - \lambda)$ for all $\lambda \in \mathbb{C}$.

Proof. It is obvious from Theorem 2.6.

Theorem 2.9. If $T \in Q(A_n^*)$ and $r, s \in \dagger_p(T)$ with $r \neq s$, then $\ker(T - r) \perp \ker(T - s)$.

Proof. Let $x \in \ker(T - r)$ and $y \in \ker(T - s)$. Then $Tx = rx$ and $Ty = sy$. Therefore

$$r\langle x, y \rangle = \langle Tx, y \rangle = \langle x, T^*y \rangle = \overline{s}\langle x, y \rangle,$$

then $\langle x, y \rangle = 0$. Therefore, $\ker(T - r) \perp \ker(T - s)$.

Theorem 2.10. If $T \in Q(A_n^*)$ has the representation $T = \lambda \oplus A$ on $\ker(T - \lambda) \oplus \ker(T - \lambda)^\perp$, where λ is an eigenvalue of T , then A belongs to class $Q(A_n^*)$, with $\ker(A - \lambda) = \{0\}$.

Proof. Since $T = \lambda \oplus A$, then $T = \begin{pmatrix} \lambda & 0 \\ 0 & A \end{pmatrix}$ and we have

$$T^*|T^{n+1}|^{\frac{2}{n+1}}T - T^*|T^*|^2T = \begin{pmatrix} \lambda^4 & 0 \\ 0 & A^*|A^{n+1}|^{\frac{2}{n+1}}A \end{pmatrix} - \begin{pmatrix} \lambda^4 & 0 \\ 0 & A^*|A^*|^2A \end{pmatrix} = \begin{pmatrix} \lambda^4 & 0 \\ 0 & A^*|A^{n+1}|^{\frac{2}{n+1}}A - A^*|A^*|^2A \end{pmatrix}$$

Since $T \in Q(A_n^*)$, then $A \in Q(A_n^*)$.

Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \ker(A - \lambda)$, then

$$(T - \lambda) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & A - \lambda \end{pmatrix} \begin{pmatrix} 0 \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

so $x \in \ker(T - \lambda)$, therefore $x=0$.

Lemma 2.11. [5, Holder-McCarthy inequality] Let T be a positive operator. Then the following inequalities hold for all $x \in H$

- 1). $\langle T^r x, x \rangle \leq \langle Tx, x \rangle^r \|x\|^{2(1-r)}$ for $0 < r < 1$,
- 2). $\langle T^r x, x \rangle \geq \langle Tx, x \rangle^r \|x\|^{2(1-r)}$ for $r \geq 1$.

A complex number λ is said to be in the approximate point spectrum $\dagger_a(T)$ of T if there is a sequence $\{x_n\}$ of unit vectors satisfying $(T - \lambda)x_n \rightarrow 0$. If in additions $(T - \lambda)^*x_n \rightarrow 0$ then λ is said to be in the joint approximate point spectrum $\dagger_{ja}(T)$ of operator T . Clearly $\dagger_{ja}(T) \subseteq \dagger_a(T)$. In general $\dagger_{ja}(T) \neq \dagger_a(T)$.

There are many classes of operators for which

$$\dagger_{ja}(T) = \dagger_a(T) \quad (2)$$

for example, if T is either normal or hyponormal operator. In [25] Xia showed that if T is a semihyponormal operator then holds (2). Dugall et.al extended this result to $*$ -paranormal operators in [10]. Cho and Yamazaki in [7] showed that if T is class A operator, then

nonzero points of $\dagger_{ja}(T)$ and $\dagger_a(T)$ are identical. In the following, we will show that if $T \in Q(A_n^*)$, then nonzero points of $\dagger_{ja}(T)$ and $\dagger_a(T)$ are identical.

Theorem 2.12. If T is of the class $Q(A_n^*)$ operator, and $(T - \lambda)x_m \rightarrow 0$ for $\lambda \neq 0$, then $(T - \lambda)^* x_m \rightarrow 0$.

Proof. Let T be a class $Q(A_n^*)$ operator and $\|(T - \lambda)x_m\| \rightarrow 0$. We may assume that $\|x_m\| = 1$. By the assumption $(T - \lambda)x_m \rightarrow 0$, from

$$T^k = (T - \lambda + \lambda)^k = \sum_{i=1}^k \binom{k}{i} \lambda^{k-i} (T - \lambda)^i + \lambda^k, \text{ for } k \in \mathbb{N},$$

we have $(T^{2+n} - \lambda^{2+n})x_m \rightarrow 0$.

By

$$\|T^{2+n}x_m - \lambda^{2+n}\| \leq \|(T^{2+n} - \lambda^{2+n})x_m\|$$

hence

$$\|T^{2+n}x_m\| \rightarrow |\lambda|^{2+n} \quad (3)$$

Moreover

$$\left| \|T^*x_m\| - \|T^*(T - \lambda)x_m\| \right| \leq \|T^*Tx_m\|. \quad (4)$$

So

Since $T \in Q(A_n^*)$, by Holder-McCarthy inequality, we get

$$\begin{aligned} \|T^*Tx_m\|^2 &= \langle T^*|T|^2Tx_m, Tx_m \rangle \leq \langle T^*|T|^{1+n}|T|^{\frac{2}{1+n}}Tx_m, Tx_m \rangle \leq \\ &\langle |T|^{1+n}|T|^2Tx_m, Tx_m \rangle^{\frac{1}{1+n}} \|Tx_m\|^{\frac{2n}{n+1}} = \|T^{2+n}x_m\|^{\frac{2}{1+n}} \|Tx_m\|^{\frac{2n}{n+1}} \end{aligned}$$

So,

$$\|T^*Tx_m\| \leq \|T^{2+n}x_m\|^{\frac{1}{1+n}} \|Tx_m\|^{\frac{n}{n+1}}$$

Then it follows from (3), (4) and (5) that

$$\limsup_{m \rightarrow \infty} \|T^*x_m\| \leq |\lambda|.$$

Since

$$\begin{aligned} \|(T - \lambda)^*x_m\|^2 &= \langle T^*x_m, T^*x_m \rangle - \overline{\lambda} \langle x_m, T^*x_m \rangle - \lambda \langle T^*x_m, x_m \rangle + |\lambda|^2 = \\ \|T^*x_m\|^2 - \overline{\lambda} \langle Tx_m, x_m \rangle - \lambda \langle x_m, Tx_m \rangle + |\lambda|^2 &= \|T^*x_m\|^2 - \overline{\lambda} \langle (T - \lambda)x_m, x_m \rangle - \lambda \langle x_m, (T - \lambda)x_m \rangle + |\lambda|^2 \end{aligned}$$

then

$$\limsup_{m \rightarrow \infty} \|(T - \lambda)^*x_m\|^2 \leq |\lambda|^2 - |\lambda|^2 = 0.$$

This implies $(T - \lambda)^*x_m \rightarrow 0$.

Corollary 2.13. If T is of the class $Q(A_n^*)$ operator, then $\dagger_{ja}(T) \setminus \{0\} = \dagger_a(T) \setminus \{0\}$.

Lemma 2.14.[3] Let $T = U|T|$ be the polar decomposition of T , $\lambda \neq 0$ and $\{x_m\}$ a sequence of vectors. Then the following assertions are equivalent:

- 1). $(T - \lambda)x_m \rightarrow 0$ and $(T^* - \bar{\lambda})x_m \rightarrow 0$,
- 2). $(|T| - |\lambda|)x_m \rightarrow 0$ and $(U - e^{i\lambda})x_m \rightarrow 0$,
- 3). $(|T^*| - |\lambda|)x_m \rightarrow 0$ and $(U^* - e^{-i\lambda})x_m \rightarrow 0$.

Theorem 2.15. If T is of the class $Q(A_n^*)$ operator and $\lambda \in \rho_a(T)$, $\{0\}$ then $|\lambda| \in \rho_a(|T|) \cap \rho_a(|T^*|)$.

Proof. If $\lambda \in \rho_a(T)$, $\{0\}$, then by Theorem 2.12., there exists a sequence of unit vectors $\{x_m\}$ such that $(T - \lambda)x_m \rightarrow 0$ and $(T - \lambda)^*x_m \rightarrow 0$. Hence, from Lemma 2.14. we have $|\lambda| \in \rho_a(|T|) \cap \rho_a(|T^*|)$.

Let $Hol(\rho_a(T))$ be the space of all analytic functions in an open neighborhood of $\rho_a(T)$. We say that $T \in L(H)$ has the single valued extension property at $\lambda \in \rho_a(T)$, if for every open neighborhood U of λ the only analytic function $f: U \rightarrow H$ which satisfies equation $(T - \lambda)f(\lambda) = 0$, is the constant function $f \equiv 0$. The operator T is said to have SVEP if T has SVEP at every $\lambda \in \rho_a(T)$. An operator $T \in L(H)$ has SVEP at every point of the resolvent $\rho(T) = \rho_a(T) \cup \rho_s(T)$. Every operator T has SVEP at an isolated point of the spectrum.

For $T \in L(H)$, the smallest nonnegative integer p such that $\ker T^p = \ker T^{p+1}$ is called the ascent of T and is denoted by $p(T)$. If no such integer exists, we set $p(T) = \infty$. We say that $T \in L(H)$ is of finite ascent (finitely ascensive) if $p(T - \lambda) < \infty$, for all $\lambda \in \rho_a(T)$.

Corollary 2.16. If T is of the class $Q(A_n^*)$ operator, then $T - \lambda$ has finite ascent for all $\lambda \in \rho_a(T)$.

Proof. We have to tell that $\ker(T - \lambda) = \ker(T - \lambda)^2$. To do that, it is sufficient enough to show that $\ker(T - \lambda)^2 \subseteq \ker(T - \lambda)$, since $\ker(T - \lambda) \subseteq \ker(T - \lambda)^2$ is clear.

Let $x \in \ker(T - \lambda)^2$, then $(T - \lambda)^2x = 0$. From Theorem 2.6. we have $(T - \lambda)^*(T - \lambda)x = 0$. Hence,

$$\|(T - \lambda)x\|^2 = \langle (T - \lambda)^*(T - \lambda)x, x \rangle = 0,$$

so we have $(T - \lambda)x = 0$, which implies $\ker(T - \lambda)^2 \subseteq \ker(T - \lambda)$.

Corollary 2.17. If T is of the class $Q(A_n^*)$ operator, then T has SVEP.

Proof. Proof, obvious from [2, Theorem 2.39].

An operator $T \in L(H)$ is called an upper semi-Fredholm, if it has a closed range and $r(T) < \infty$, while T is called a lower semi-Fredholm if $s(T) < \infty$. However, T is called a semi-Fredholm operator if T is either an upper or a lower semi-Fredholm, and T is said to be a Fredholm operator if it is both an upper and a lower semi-Fredholm. If $T \in L(H)$ is semi-Fredholm, then the index is defined by

$$\text{ind}(T) = r(T) - s(T).$$

An operator $T \in L(H)$ is said to be upper semi-Weyl operator if it is upper semi-Fredholm and $\text{ind}(T) \leq 0$, while T is said to be lower semi-Weyl operator if it is lower semi-Fredholm and $\text{ind}(T) \geq 0$. An operator is said to be Weyl operator if it is Fredholm of index zero.

The Weyl spectrum and the essential approximate spectrum are defined by

$$\rho_w(T) = \{\lambda \in \rho(T) : T - \lambda \text{ is not Weyl}\},$$

and

$$\rho_{uw}(T) = \{\lambda \in \rho(T) : T - \lambda \text{ is not upper semi-Weyl}\}.$$

An operator $T \in L(H)$ is said to be upper semi-Browder operator, if it is upper semi-Fredholm and $p(T) < \infty$. An operator $T \in L(H)$ is said to be lower semi-Browder operator, if it is lower semi-Fredholm and $q(T) < \infty$. An operator $T \in L(H)$ is said to be Browder operator, if it is Fredholm of finite ascent and descent. The Browder spectrum and the upper semi-Browder spectrum are defined by

$$\dagger_b(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not Browder} \},$$

and

$$\dagger_{ub}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not upper semi-Browder} \}.$$

Theorem 2.18. If T or T^* belongs to class $Q(A_n^*)$, then $\dagger_w(f(T)) = f(\dagger_w(T))$ for all $f \in Hol(\dagger(T))$.

Proof. The inclusion $f(\dagger_w(T)) \subseteq \dagger_w(f(T))$ holds for any operator. Let $T \in Q(A_n^*)$, then T has SVEP, then from [2, Theorem 4.19] holds $\dagger_w(f(T)) \subseteq f(\dagger_w(T))$. If $T^* \in Q(A_n^*)$, similar to above.

The following concept has been introduced in 1997 by Harte and W.Y. Lee [16]: A bounded operator T is said to satisfy Browder's theorem if $\dagger_w(T) = \dagger_b(T)$.

Theorem 2.19. If T or T^* belongs to class $Q(A_n^*)$, then $f(T)$ satisfy Browder's theorem for all $f \in Hol(\dagger(T))$.

Proof. Since T or T^* has SVEP, then from [2, Theorem 4.22] $f(T)$ satisfies Browder's theorem for all $f \in Hol(\dagger(T))$.

The following concept has been introduced in 2000, [8]: A bounded operator T is said to satisfy a-Browder's theorem if $\dagger_{uw}(T) = \dagger_{ub}(T)$.

Theorem 2.20. If T or T^* belongs to class $Q(A_n^*)$, then a-Browder's theorem holds for $f(T)$ and $f(T)^*$ for all $f \in Hol(\dagger(T))$.

Proof. Since T or T^* has SVEP, then from [2, Theorem 4.33] $f(T)$ and $f(T)^*$ satisfies Browder's theorem for all $f \in Hol(\dagger(T))$.

3. Tensor products for quasi *-class A_n

Let H and K denote the Hilbert spaces. For given non zero operators $T \in L(H)$ and $S \in L(K)$, $T \otimes S$ denotes the tensor product on the product space $H \otimes K$. The normaloid property is invariant under tensor products, [22]. There exist paranormal operators T and S , such that $T \otimes S$ is not paranormal, [1]. In [23], Stochel proved $T \otimes S$ is normal, if and only if, T and S are normal. This result was extended to class A operators, *-class A operators, class A_n operators and quasi class A_n operators in [18], [10], [20], and [21] respectively. In this section, we prove an analogues result for $Q(A_n^*)$ operators.

Lemma 3.1. [9] Let $T \in L(H)$ and $S \in L(K)$ be non zero operators. Then:

- 1). $(T \otimes S)^*(T \otimes S) = T^*T \otimes S^*S$,
- 2). $\|T \otimes S\|^p = \|T\|^p \otimes \|S\|^p$ for any positive real number p .

Theorem 3.2. Let $T \in L(H)$ and $S \in L(K)$ be non zero operators. Then $T \otimes S$ belongs to $Q(A_n^*)$ operator, if and only if, one of the following holds:

- 1). T and S are $Q(A_n^*)$,

2). $T^2 = 0$ or $S^2 = 0$.

Proof. We have

$$\begin{aligned} & (T \otimes S)^* \left(| (T \otimes S)^{n+1} |^{\frac{2}{n+1}} - | (T \otimes S)^* |^2 \right) (T \otimes S) = \\ & (T \otimes S)^* \left(| T^{n+1} |^{\frac{2}{n+1}} \otimes | S^{n+1} |^{\frac{2}{n+1}} - | T^* |^2 \otimes | S^* |^2 \right) (T \otimes S) = \\ & T^* \left(| T^{n+1} |^{\frac{2}{n+1}} - | T^* |^2 \right) \otimes S^* | S^{n+1} |^{\frac{2}{n+1}} S + T^* | T^* |^2 T \otimes S^* \left(| S^{n+1} |^{\frac{2}{n+1}} - | S^* |^2 \right) S \end{aligned}$$

Hence, if either (1) T and S are $Q(A_n^*)$ or (2) $T^2 = 0$ or $S^2 = 0$, then $T \otimes S \in Q(A_n^*)$.

Conversely, suppose that $T \otimes S$ is a $Q(A_n^*)$ operator. Then, for $x \in H$, $y \in K$ we gain

$$\left\langle T^* \left(| T^{n+1} |^{\frac{2}{n+1}} - | T^* |^2 \right) T x, x \right\rangle \left\langle S^* | S^{n+1} |^{\frac{2}{n+1}} S y, y \right\rangle + \left\langle T^* | T^* |^2 T x, x \right\rangle \left\langle (S^* | S^{n+1} |^{\frac{2}{n+1}} - | S^* |^2) S y, y \right\rangle \geq 0$$

It suffices to show that if the statement (1) does not hold, then the statement (2) holds. Thus, assume to the contrary that neither T^2 nor S^2 is the zero operator, and T is not $Q(A_n^*)$ operator. Then, there exists $x_0 \in H$, such that:

$$\left\langle T^* \left(| T^{n+1} |^{\frac{2}{n+1}} - | T^* |^2 \right) T x_0, x_0 \right\rangle = r < 0 \text{ and } \left\langle T^* | T^* |^2 T x_0, x_0 \right\rangle = s > 0.$$

From above relation, we have

$$(r + s) \left\langle S^* | S^{n+1} |^{\frac{2}{n+1}} S y, y \right\rangle \geq s \left\langle S^* | S^* |^2 S y, y \right\rangle.$$

Thus, $S \in Q(A^*)$ operator, because $r + s < s$.

We have,

$$\left\langle S^* | S^* |^2 S y, y \right\rangle = \left\langle | S^* |^2 S y, S y \right\rangle = \left\langle S^* S y, S^* S y \right\rangle = \| S^* S y \|^2$$

and using the Holder McCarthy inequality, we get

$$\left\langle S^* | S^{n+1} |^{\frac{2}{n+1}} S y, y \right\rangle = \left\langle (S^{*(n+1)} S^{n+1})^{\frac{1}{n+1}} S y, S y \right\rangle \leq \left\langle S^{*(n+1)} S^{n+1} S y, S y \right\rangle^{\frac{1}{n+1}} \| S y \|^{\frac{2n}{n+1}} = \| S^{n+2} y \|^{\frac{2}{n+1}} \| S y \|^{\frac{2n}{n+1}}$$

Then ,

$$(r + s) \| S^{n+2} y \|^{\frac{2}{n+1}} \| S y \|^{\frac{2n}{n+1}} \geq s \| S^* S y \|^2 .$$

Since $S \in Q(A_n^*)$, from Theorem 2.3. S has decomposition of the form

$$S = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \text{ on } H = \overline{S(H)} \oplus \ker S^*$$

where $A = S|_{\overline{S(H)}}$ is A_n^* operator, we have

$$(r + s) \| A^{n+2} \sim \|^{\frac{2}{n+1}} \| A \sim \|^{\frac{2n}{n+1}} \geq s \| A^* A \sim \|^2 ,$$

for all $\sim \in \overline{S(H)}$.

Since $A \in A_n^*$, then A is normaloid, since A is k -*-paranormal operator. Thus, taking supremum on both sides of the above inequality, we have

$$(r + s) \|A\|^4 \geq s \|A\|^4.$$

This inequality makes $A=0$. Hence, $S^2 = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}^2 = 0$. This is a contradiction to that S^2 is not a zero operator. So T must be a $Q(A_n^*)$ operator. A similar argument shows that S is also a $Q(A_n^*)$ operator, which completes the proof.

References

- [1] T. Ando, Operators with a norm condition, *Acta Sci. Math. (Szeged)*33(1972), 169-178.
- [2] P. Aiena, Semi-Fredholm operators, perturbations theory and localized SVEP, Merida, Venezuela 2007.
- [3] A. Aluthge and D. Wang, The joint approximate point spectrum of an operator, *Hokkaido Mathe J.*, 31(2002), 187-197.
- [4] S.C. Arora, J.K. Thukral, On a class of operators, *Glas. Math.* 21 (1986) 381–386.
- [5] C.A.McCarthy, *Cp, Israel J.Math.*, 5.(1967) 249-271.
- [6] N. Chennappan, S. Karthikeyan, *-Paranormal composition operators, *Indian J. Pure Appl. Math.*, 31(6), (2000), 591-601.
- [7] M. Cho and T. Yamazaki, An operator transform from class A to the class of hyponormal operators and its application, *Integral Equations and Operator Theory*, vol. 53, no. 4, pp. 497–508, 2005.
- [8] S. V. Djordjevic and Y.M.Han, Browder's theorem and spectral continuity, *Glasgow Math. J.* 42 (2000), no.3, 479-486.
- [9] B.P. Duggal and C.S. Kubrusly, A note on k -paranormal operators, *Oper. Matrices*,4(2010), 213-223.
- [10] B. P. Dugall, I. H. Jeon, and I. H. Kim, On *-paranormal contractions and properties for *-class A operators, *Linear Algebra Appl.* 436 (2012), no. 5, 954-962.
- [11] T. Furuta, On The Class of Paranormal Operators, *Proc. Jap. Acad.* 43(1967), 594-598.
- [12] T. Furuta, M. Ito and T. Yamazaki, A subclass of paranormal operators including class of log-hyponormal and several classes, *Sci. Math.* 1 (1998), no. 3, 389–403.
- [13] T.Furuta and J.Haketa, Applications of norm inequalities equivalent to Lowner-Heinz theorem, *Nihonkai J. Math.* 1 (1990), 11-17.
- [14] F. Hansen, An operator inequality, *Math. Ann.* 246 (1980) 249-250.
- [15] J. K. Han, H. Y. Lee, and W. Y. Lee, Invertible completions of 2×2 upper triangular operator matrices, *Proceedings of the American Mathematical Society*, vol. 128, no. 1, pp. 119-123, 2000.
- [16] R. E. Harte and W. Y. Lee, Another note on Weyl's theorem, *Trans.Amer. Math.Soc.* 349. No.1 2115-2224.
- [17] Shen Li Jun, Zuo Fei and Yang Chang Sen, On Operators Satisfying $T^*|T^2|T \geq T^*|T^*|^2T$, *Acta Mathematica Sinica, English Series*, Nov., 2010, Vol. 26, No. 11, pp. 2109-2116.
- [18] I. H. Kim, Weyl's theorem and tensor product for operators satisfying $T^{*k}|T^2|T^k \geq T^{*k}|T^*|^2T^k$, *J. Korean Math. Soc.*, 47(2010), No. 2, 351-361.

- [19] F. Kimura, Analysis of non-normal operators via Aluthge transformation, *Integral Equations and Operator Theory* 50 (3)(1995), 375–384.
- [20] S.Panayappan, N. Jayanthi, D.Sumathi, Weyl’s theorem and tensor product for Class $A_{\{k\}}$ operators, *Pure Mathematical Sciences*, Vol. 1 (2012), no.1, 13-23.
- [21] S. Panayappan, N. Jayanthi, D. Sumathi, Weyl’s theorem and tensor product for quasi Class A_k operators}, *Pure Mathematical Sciences*, Vol. 1 (2012), no.1, 33-41.
- [22] T. Saito, Hyponormal operators and Related topics, *Lecture notes in Mathematics*, Springer-Verlag, 247(1971).
- [23] J. Stochel, Seminormality of operators from their tensor products, *Proc. Amer. Math.*, 124(1996), 435-440.
- [24] A. Uchiyama, Weyl’s theorem for class A operators, *Mathematical Inequalities and Applications*, vol.4, no. 1, pp. 143–150, 2001.
- [25] D. Xia, *Spectral Theory of Hyponormal Operators*, Birkhauser, Switzerland 1983.